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APPLIED MECHANICS

WOODWARD

FIRST YEAR WORK

IN

APPLIED MECHANICS

BY

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PREFACE.

This book is written primarily for students entering upon their second collegiate year. Their knowledge of mechanics is limited to what was gained during their study of physics and from practice in laboratory and shop; hence no apology is needed for making matters plain and easy in the first chapters of a work which later on leads up to the Theory of Structures, Internal Stress, Motion under Complicated Conditions, Forms of Energy, and higher Graphics.

It is assumed that at every stage of progress the student can by reasonable effort read the book understandingly *by himself* (always with pencil and paper at hand), so as to be ready to grasp the conditions of concrete or ideal problems brought up in class, and follow statements made in the language of mathematics and mechanics. In other words, he must here learn to translate the language of the world of matter, motion, and force into mathematical forms.

The book is not flooded with easy examples which involve nothing new. The use of Graphical solutions is a characteristic feature. The infinitesimal Calculus is not introduced until Centroids and Centers of Action are reached in Chapter VII. In Chapter IX, fiber stress in a Bent Beam is introduced in order to fully explain the origin and use of the Moment of Inertia of surfaces. The comprehensive study of beams will follow in Chapters XIX-XXI.

The comparatively easy matter of direct Stresses in Framed Structures belongs in one's first year in Mechanics. Static Triangles and Static Polygons are preferred to abstract equations, tho the latter are not excluded.

The titles of subsequent chapters sufficiently indicate their content.

The entire volume will be ready for delivery before Christmas, 1912.

Notice of errors, whether by the author or the printer, will be thankfully received.

3013 Hawthorne Bl.,
St. Louis, Sept., 1912.

CALVIN M. WOODWARD.

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APPLIED MECHANICS.

CHAPTER I.

Introductory.

1. Force is an action between two bodies. It is not a tendency, it is a real thing as experience readily shows. A pushes or pulls B, and at the same time B pushes or pulls A; there are two bodies and one action or force. There can be no force unless there be at least two bodies. The bodies may be at rest or they may move. There may be an action, a push or pressure, between two bricks in a wall which does not move. There may be a pull or tension between two cars which are in motion.

1. In Applied Mechanics the general surface of the earth is assumed to be at rest. In Celestial Mechanics full account is taken of the motions of the earth and of other heavenly bodies.

All forces are more or less distributed, either over surfaces or thru volumes. The pressure between one's foot and the floor, or between steam and a piston, is distributed over the surface of contact; in one case unevenly, in the other uniformly; in one case without motion, in the other with motion. The earth, in a most mysterious way, pulls upon every particle of matter:—an apple hanging on a tree, a brick in a wall, and an iron ball flying thru the air; and of necessity the apple, the brick, and the ball pull the earth. This is called the "force of gravitation." Of all the forces with which mechanics deals, gravitation is the most common, and the most inexplicable. We know, however, that nothing can escape it, and that it is not sensibly affected by such distances as are generally considered in this book, or by intervening objects.

2. It is highly important that the student regards the force of gravitation as a pull, distributed thruout the whole volume of a body. Moreover the action (or pull) of gravitation upon a body must not be confused with the action between the body and the platform or foundation upon which the body may rest. When a body like a block of stone rests upon the ground, there are two actions between the stone and the earth, viz: a pull down and a push up, which exactly balance, one action being distributed thru volumes, the other being distributed over the surface of contact.

3. The plural word "volumes" is used advisedly. One volume is that of the stone which is made up of an infinite number of particles;

the other volume is that of the earth which also consists of an infinite number of particles; and there is an action between every particle in the stone and every particle in the earth. What we call the *weight* of a body is the *resultant* of all the separate pulls, and its direction is towards and away from the earth's center. The stone pulls the earth *just as much* as the earth pulls the stone. See "ATTRACTION" in a later Chapter.

2. When Sir Isaac Newton, the master mind in Mechanics, stated as *a fundamental law that action and reaction were equal and opposite*, he meant only this: That when there is an action between A and B, the measure of A's action on B is exactly equal to the measure of B's action on A, and in the opposite direction. This sounds as though there were two forces for one action, but such is not the case. There is only one force, but its action may be viewed from the standpoint of A, or from the standpoint of B. That is, if we are considering the condition or behavior of A as regards rest or motion, we only take into account the action of B upon it, and we give that action its proper direction; if, however, we are considering the status of B, we take account only of A's action upon it.

1. This law of Newton does not explain the stability of the block of stone which rests on the ground in a former illustration. In that case there were two actions between the earth and the stone, a pull and a push, that is, an attraction and a pressure; and the pull and push balanced.

It is true in the case considered, that if there were no pull, there would be no push, because we have assumed that the stone was at rest, and hence the two actions must balance. If there was but one action, the stone would not be at rest; this must be made clear.

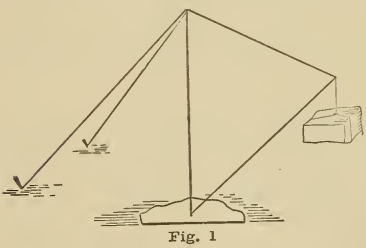


Fig. 1

2. Suppose a stone to be suspended by a derrick, and to be at rest. There is now an action between the stone and the earth; the earth is pulling *down* on the stone and the stone is pulling *up* on the earth; that is one case of action. In the next place there is an action between the stone and the derrick; the stone is pulling down on the derrick, and the derrick is pulling up on the stone—this is a second case of action. Thus we see that the stone is acted upon by two bodies: the earth which pulls down and the derrick which pulls up, and as the stone is at rest we know that these two pulls are equal in magnitude and exactly opposite.

Now suppose the rope or chain thru which the derrick acts, is cut or breaks. The action between the stone and the derrick ceases, and only the action between the stone and the earth remains. Hence it cannot be at rest; and the stone and the earth begin at once to approach each other. However, the earth's motion is so immeasurably small that we leave it wholly out of account, and merely say: "the stone *falls*."

3. Stress. Thus far we have thought of A and B as two separate bodies, but it is evident that they may be separate parts of the same body like adjacent strata in the earth, adjacent leaves in a book, adjacent links in a chain, adjacent particles in a steel rod or a concrete post. When A and B are adjacent parts of a continuous body, their mutual action, no matter what its character may be, is called *Stress*. If it be a pull it is called *tensile stress*, and the continuous body, rope, wire, rod or bar is said to be in *tension*. If the adjacent layers press against or upon each other in a post, strut, block or wall, the post, strut, block or wall is in compression, and the stress is called *compressive stress*.

The link of a chain illustrates fairly well adjacent parts of a continuous body. Suppose in Fig. (1) that the stone is suspended to the derrick boom by means of a chain. The chain is in tension and every link is acted upon by two forces (independently of the earth's pull or attraction upon the material in the link) the downward pull of the link below it, and the upward pull of the link above it. Furthermore, it is evident that the upward pull of the topmost link must equal in magnitude the downward pull of the lowest link, plus the pull, or attraction, of the earth upon all the material in the intermediate links.

4. Magnitude of forces. Units. We are immediately conscious that forces or actions have magnitude; one pull or attraction is greater or less than another; one push, pressure or repulsion is greater or less than another. In order to express magnitude with precision we must have a well known unit of force with which all other forces may be compared numerically. A certain pull or push shall be called *one*, and like other units in common use, it shall have a name. The unit of time in mechanics the world over is a *second* (or a multiple of a second); the unit of length or distance most commonly used in the Anglo-Saxon countries is a *foot* (or a multiple thereof), though the meter, a French unit, is in common use in physical text-books and laboratories. The unit of value in North America is a *dollar*. In all these cases we know by observation, experience and frequent use just what these units are, and what a given number of seconds, feet, meters or dollars means.

5. Units of force. When, however, we come to the unit of force and its name, we find great diversity in the text books, lecture rooms, and laboratories of every engineering school in the land, in spite of all efforts of the users to bring about the adoption and general use of a common unit.

1. The first and most common is the *avoirdupois pound* by which we measure the weights of all sorts of things, building materials, foods, crops, ores, manufactured articles, etc. (*i. e.*, the pull of the earth upon them), the tensions in ropes, cords, chains, drawbars, rods and eye-bars; and the pressures of gases (air, steam, etc.), and of liquids (water, oils, etc.).

2. The second unit of force is the *poundal*. This unit of force is about one-thirty-second of an avoirdupois pound, or half an ounce. Its use is extremely limited, but is not unfrequently found in works on applied electricity, where it measures the electrical forces of attraction and repulsion.

3. The third unit of force is a *kilogram*, or its one-thousandth part called the *dyne*. This unit of force is of French origin, but its use, like that of the meter, is world-wide in physics, chemistry and engineering. In magnitude the Kilogram is about 2.2 pounds.

6. The meter (which was intended to be one ten-millionth of a quadrant of the earth's meridian, and which is approximately so), is really the distance between two engraved marks on a platinum-iridium bar, carefully preserved at Paris. An accurate copy of this bar is in the archives at Washington. The meter bar was received by the President of the United States in 1890; the entire metric system is now legalized in all the territory under our flag. The *foot*, consisting of 12 inches, is defined as a certain part of a *meter*, shown by the relation that

$$1 \text{ meter} = 39.37 \text{ inches}.$$

1. At the same time in 1890 a standard block of metal (platinum-iridium) was received and deposited at Washington, which, at the level of the sea, and at 45° north latitude was declared to weigh one *kilogram* (kg.). Another block, under the same conditions, is declared to weigh one *pound* (lb.). The relation is very exactly

$$\text{One kilogram} = 2.20462 \text{ lbs}.$$

2. The relation of the kilogram to the meter is shown by this: A kilogram is the weight, at the level of the sea, latitude 45°, of one one-thousandth part of a cubic meter of distilled water at a temperature of 0° centegrade, or 32° Fahrenheit. Accordingly, a cubic meter of water weighs 1,000 kgs. under standard conditions.

3. It will be well for the student to fix these standard relations well in his mind as he must think readily in terms of each.

1 meter = 39.37 inches.
1 foot = 0.3048 meter.
1,000 meters = 1 kilo-meter = $\frac{5}{8}$ mile, nearly = 3281 feet.
1 kilogram = about 2.2 lbs.
1 ton = 2,000 lbs. = 907.2 kgs.

For greater precision (rarely necessary) see an encyclopedia.

7. Units of mass and time. There are in every system of units a unit of *time* and a unit of *mass*, which will be fully explained when we come to expound the laws of motion and moving bodies.

Meanwhile, the student must not be confused or disturbed by the use in text books and scientific journals of such expressions as "a mass of so many pounds," or "a mass of so many kilograms"; the meaning merely is:—a quantity of material upon which the earth's attraction is so many pounds weight, or so many kilograms. The simplest way to find the magnitude of the earth's pull or attraction upon a given quantity or mass of material is to weigh it by a spring balance, a process which need not be explained. The essential thing is to bear constantly in mind that the earth's pull upon a mass is always a *force*, and that that force should never be confused with the material pulled or acted upon.

8. Mechanical problems. 1. Problems in mechanics are always given under conditions which are more or less ideal, *i. e.*, not real. For instance, in the case of the stone hanging upon the boom of a derrick, it was assumed that the action of the air upon the stone was either nothing at all, or that its action was self-balanced, and yet we know that such could not be the fact. We know that the pressure of the air upon every square foot of the surface of the stone is about a ton. We also know that the upward pressure of the air upon the bottom or lower surface is a little greater than the downward pressure upon the top or upper surface; consequently the total up-and-down action of the air is not self-balanced. The difference or buoyancy is, however, so small compared with the other forces acting (the pull of the earth and the pull of the rope or chain), that it was neglected, and the magnitude of the earth's pull was assumed to be equal to the pull of the rope.

2. Again, there are always air currents, which are ignored, unless those currents are very strong, as in the case of high winds. In like manner we shall often assume that solids and gases are of uniform density tho we know that they always vary more or less. We shall at times suppose that the surfaces of solid bodies are *smooth*,

tho we know none that are perfectly smooth; and that the earth's pull on a body when a few feet above the sea level, or a few feet below it, is just the same as it is at sea level; and that change of latitude has no effect upon the earth's attraction, tho it is well known that strictly speaking such is not the case. Sometimes we shall assume that bodies are perfectly rigid tho we are positive that no bodies are strictly rigid.

3. Such false assumptions do not render the problems and their solutions useless. The solutions are just as accurate as are the assumptions. If the latter were sufficiently accurate, or "near enough," then the solution is near enough. Both the student and the teacher, however, are warned against the danger of ignoring conditions which are so far from *real* conditions, that the results of solutions are very misleading and mischievous. It is rarely wise to ignore friction, internal stresses and elasticity. Even with the best assumed conditions, problems are still somewhat ideal. In the problems which are solved or given for solution in this book, care will be taken to have the assumed conditions approximate closely real conditions, so that the conclusions reached may have value, and serve as guides for future conduct and use.

9. The recognition of forces. Fig. 2 should represent a heavy block resting on a table. This block supports a smaller block and a man who holds a loaded basket on his head. Partial support is given to the large block by a "flexible and weightless" rope passing round the large block at the lower end and after going over a "smooth" peg, sustains a weight of 40 lbs. at the upper end. The student is to give the *number*, *character*, "*sense*" and *magnitude* of the forces or actions upon each individual body: *A*, *B*, *C*, *D*, *E* and *P*. *Character* shows whether it is a push or a pull, and how it is distributed. "Sense" tells the direction of the action of a force on the body just then under consideration.

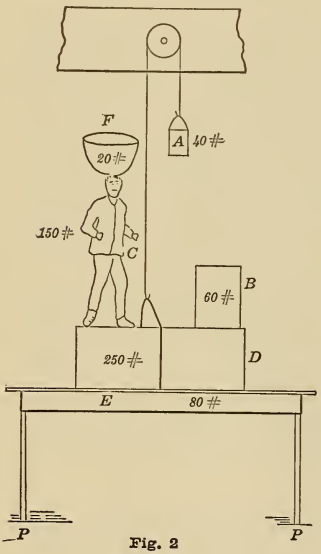


Fig. 2

For example, the Man is acted upon by three forces:

1. The *downward pull* of the earth's attraction of 150 lbs., which is *distributed* thru the *volume* of his entire person.

2. A *downward pressure* of 20 lbs. from the loaded basket *distributed* over the upper *surface* of his head.

3. An *upward pressure* of 170 lbs. from the large block *distributed* under the soles of his shoes.

The student should write out similar statements for the other bodies in the problem, paying special attention to his language as well as to his thought.

10. Ideal conditions. Several matters will be noticed which require mention. We have assumed several conditions which are more or less erroneous.

1. We have left the air entirely out of consideration whatever its action may be.

2. We have assumed that the peg over which the rope passes is perfectly smooth.

3. That the rope itself, tho strong, is perfectly flexible and imponderable, so that the tension in it is the same (viz.: 40 lbs.) from end to end.

In so-called practical problems, it is generally said that such trifles are unimportant, but it is easy to see that they might not be trifling or unimportant.

4. Finally, we must *infer* some things which the statement of the problem does not give. This inference is the result of a process of reasoning of which one is hardly conscious. How did we know that the large block pressed or lifted up against the man's shoes with a force of 170 lbs.? Had both man and block been either descending or ascending faster and faster, the action between shoes and block would not have been 170 lbs., but since both were *standing still*, we know that the forces acting on the man *must balance*. Hence the only upward force must alone be equal to the sum of the two downward forces.

5. Such reasoning we call "Common Sense," but it is strictly in accordance with the axioms or laws of mechanics. It will be seen later on that the range of common sense can be greatly increased as the laws and conditions which obtain in the interaction of bodies, at rest and in motion, are known and understood.

11. The transmission of forces. In the case of a chain under tension, it is easy to see how a pull upon the link at one end is transmitted, or *passed along*, thru all the links to the other end.

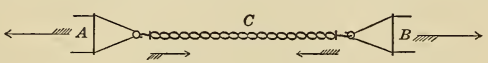


Fig. 3

Accordingly, we say that there is *an action between A and B by means of the chain C*. Still more simply, we may say, when we are considering the status of *B*, "*A* is pulling *B*." It is evident that the length of the chain is not

of any account so long as the chain, like the rope in the problem, is without weight, and therefore straight.

In like manner a body A may act upon or push a body B, by means of a strut, post, brace or leg. We shall see later that a force can be transferred or transmitted sideways (laterally) thru solid bodies.

12. Forces acting at point. In the statement and solutions of ideal problems in mechanics, perhaps the most striking assumption is, that finite forces act at points. Now all forces in reality act at surfaces or thru volumes, yet a point has neither volume nor surface. Nevertheless the notion of *concentrating* a distributed action or force is most natural and very convenient. Common experience, and therefore common sense, convinces us that a trap door, a platform, or a warehouse floor, if sufficiently rigid, could be supported by a single prop or post, *provided it be put in the proper place.*

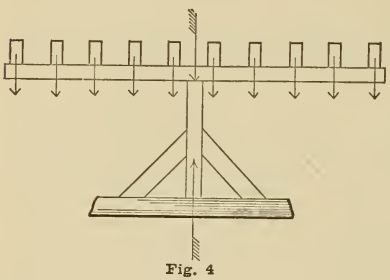


Fig. 4

Imagine a stiff plank, loaded with bricks, balanced upon the top of a small square post. This is easily done in fact, and still more easily imagined. Now the earth's attraction upon plank and bricks must be exactly balanced by the lift of the post; hence the total downward action must somehow be concentrated, or transmitted, to a point or small area just above top of the post. Similarly, the upward action of the post may be centered within a small area immediately in contact with the surface of action of the plank. These two areas may ideally be made as small as we please, *i. e.*, each may be called a "*physical point*" and the two total actions may be thought of as acting at a common point.

13. The graphical representation of a force. It will now be easy to represent an action upon a body (from any source whatever) by a straight line or arrow, the direction in which the arrow points showing the "sense" or direction of the action; the *length* of the line drawn to scale (so many pounds or tons to the foot, or so many kilograms to the meter), showing the magnitude of the force or action; and the *position* of the line showing the "line of action," any point of which, in or on the body acted upon, may be taken as the physical "point of action" of the acting body. In figure 4, the upper arrow shows the direction, magnitude and position of the earth's total attraction on the plank and bricks; and the lower arrow shows the direction, magnitude and position of the upward action of the post upon the plank.

14. Abstract forces. Altho no force can exist, except as an action between bodies, it will be convenient, when we study the concentration (composition or combination) of separate or distributed forces, to leave the description of the bodies whose actions we are considering, wholly out of our thought, and treat the forces as independent things; that is, we withdraw (or abstract) the forces from the bodies creating them, and reason about them abstractly, just as one might add six dollars to eight dollars, thereby producing fourteen dollars without having any money at all. So we may say: "Suppose we have a force fully represented by the arrow line AB, and, in the same plane with AB, another force fully represented by the arrow line CD. Our problem is now to find a single force XY whose action shall in all essentials represent, or be equal to, the combined action of the two given forces."

In the solution of this problem, we take no thought of the bodies producing the given actions. We care not what they are, just where they are, nor why they act, except that we must know their relative positions. Of course, we assume that they act upon the same rigid body, but we do not care *in this problem* what that body is. In other words our problem is purely abstract.

1. Thruout this book we shall mingle abstract problems freely with those which approximate more or less closely real ones. With abstract problems, we shall aim to illustrate general methods, establish general laws, and derive general formulas, and then apply them to the solutions of problems derived from all sorts of sources and conditions, of a practical character. The number of possible problems is infinite; many are fanciful and useless; many are too difficult for our range of mathematics; many are intensely practical and admit of easy solutions. In our analysis we shall draw freely upon the student's knowledge of Algebra, Geometry, plane and solid, and Trigonometry; and later on we shall assume Analytic Geometry and Calculus. The more thoroughly these have been mastered the better, but the more difficult operations will be clearly explained.

PART I.

STATICS.

CHAPTER II.

The Combination and Resolution of Forces.

STATICS is that department of Mechanics in which all the bodies, acting and acted upon, are at rest or stationary. A body at rest under the action of two or more other bodies is said to be in Equilibrium; and the forces acting upon it are said to Balance.

15. Forces having a common line of action. The simplest possible example is that of two forces which exactly balance or neutralize each other. It is evident without any attempt at proof that they must not only act along the same line, but they must have equal magnitudes and opposite directions. Under no other conditions can two forces balance.

1. In so far as the state of rest or motion of a rigid body is concerned, two balancing forces may be left out of account, while we are considering the actions of other forces upon the same body. This cancellation of two balancing forces, is analagous to the striking off of equal terms in the members of an algebraic equation.

2. **Conversely,** we may at any time, to facilitate a solution, introduce two balancing forces, representing the actions of two imaginary bodies upon the body we are considering, without in any way affecting its state of rest or motion.

3. When several forces, of different magnitudes, have a common line of action, while some act in one direction and some in the other, they are readily reduced to two by adding those which have a common direction, and then reducing the two to one by allowing the smaller to cancel or neutralize an equally large part of the other; or the cancellation and addition may be carried on simultaneously. All this is readily expressed in the language of algebra.

Let F be the numerical measure of the magnitude of a force, and let one direction along the common line of action be considered as positive, and the other direction negative.

ΣF represents a *process* of algebraic addition of as many numbers (of units of force) as there were of assumed forces, acting upon a (real or imaginary) body. Finally, if we represent the magnitude

of the single force which *results* from the addition by the letter R, we have our first equation in mechanics.

$$\Sigma F = R \qquad (1)$$

If the forces under consideration actually balance, the value of R is zero, and the equation becomes

$$\Sigma F = 0 \qquad (2)$$

16. Conversely. Since forces having a common line of action may be combined to form a single force, so a force may be resolved or separated into any number of component forces having the same line of action and preserving the algebraic sum.

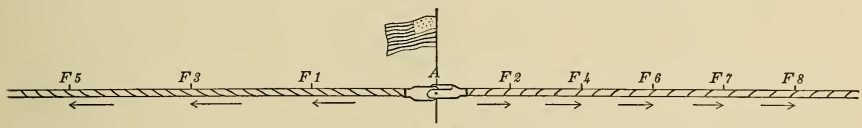


Fig. 5

The student should not fail to see the full meaning of these equations. An illustration will surely be of value. Fig. 5 is to represent a "tug of war" between a team of three men and a team of five boys. The three men have hold of one rope and are pulling to the left. The boys have another rope and they pull to the right. Both ropes are attached to a pin carrying a flag. Ropes, pin and flag are "imponderable," *i. e.*, their weight is not taken into account. Of course the effort of each team will be to pull the pin and flag its way. As soon as the signal is given to "Pull!" eight bodies (boys and men) act (thru the mediation of the ropes) upon the body (pin) A. Let us assume that forces acting towards the right are positive and those to the left are negative, and that

$F_8 = +200$ lb $\quad F_2 = +180$ lb $\quad F_4 = +175$ lb $\quad F_1 = -306$ lb
$F_3 = -284$ lb $\quad F_6 = +140$ lb $\quad F_2 = +150$ lb $\quad F_5 = -255$ lb

Now in this case ΣF means

$$+180+140+200-284+175+150-306-255$$

and since the algebraic sum is zero,

$$\Sigma F = 0.$$

Hence the forces balance, and that the pin A stands still.
If on a second pull $\Sigma F < 0$, the men win; if $\Sigma F > 0$, the boys win.

17. An action in Statics does not produce motion; on the contrary an action prevents motion. A brick in the wall is acted upon by the

brick above it, and the brick below it, and by the earth which attracts it, but it does not move, because the forces balance each other, that is: the action of the lowest brick is equal and opposite to the combined action of "gravitation" and the uppermost brick.

1. When forces do not balance, and motion is caused or modified, we have a problem in Kinetics, which will be discussed in later chapters.

18. Couples. When a body is acted upon by two other bodies in such a manner that the two forces have equal magnitudes, different but parallel lines of action, and opposite directions, they cannot balance, neither can their joint actions have a tendency to move the body up or down, to the right or to the left. Their sole tendency is to make the body *turn*, or, if it be already turning, to modify that turning.

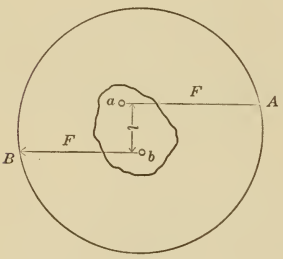


Fig. 6

1. Suppose a piece of cork to be floating upon a tub of water. The pull of the earth upon it is exactly balanced by the upward pressure of the water under it so that it is at rest, and we may ignore these two balanced forces. Now suppose that two needles are prest into the upper surface of the cork at *a* and *b*, Fig. 6, and that two parallel threads are stretched to the edge of the tub in a plane parallel to the surface of the water, as shown. Suppose that at *A* and *B* gentle but *equal* pulls upon the threads are given simultaneously. Instantly the cork begins to turn "right handed." These two new forces form a "right-hand couple," whose tendency is obvious. The perpendicular distance between the two lines of action is called the *arm* of the couple.

2. Next suppose the cork at rest again and that two more needles are put in at *c* and *d*, Fig. 7, and that threads are carried across and *fastened* at *C* and *D*, so that they are parallel to each other and in a plane parallel to the surface of the water.

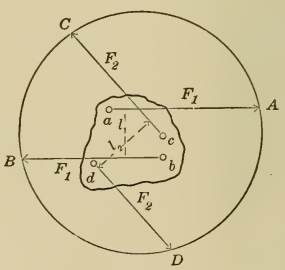


Fig. 7

If now the equal pulls at *A* and *B* are again applied, the cork stands still, and it will be found that the threads *cC* and *dD* are in tension. Moreover, since the cork does not move either towards *C* or towards *D*, the tension in *cC* and in *dD* are equal; they thus form a "left-hand couple." We consequently have two couples which balance each other. It is evident that the turning

effect or tendency of a couple must depend partly upon the *common magnitude* of the two forces, and partly upon the *distance between the two lines of action.* In short, *the magnitude of a couple's influence is measured by the product of one of its forces by its arm.* This product is called a "*moment,*" and it will be represented by the capital letter M, so that

$$M = Fl$$

This will be proved when we come to resolve a couple into Fl unit couples; **26.**

3. The student must bear in mind that F in this equation is the *number* of units of force in one of the given forces, and l is the *number* of units of length in the given distance (between the parallel lines of action), and that therefore M is the *number* of units of moment.

When we speak of "the *force* of a couple," we shall mean the common magnitude F.

19. The unit moment is not a fundamental unit, but it is derived from two which are fundamental: the foot and the pound; hence a moment is read as so many "foot-pounds, foot-tons," etc.

The word "*foot-pound*"as here used must not be mistaken for a later use of the same word as the *unit of work.* The moment unit expresses and measures the turning action of a body, but its action does not require actual motion. An *unbalanced* moment would turn a body; but in all problems in Statics, moments are balanced and no motion results.

20. The Freedom of a Balancing Couple. Referring again to Fig. 7, it is evident that the needles c and d could have been placed anywhere on the cork, and that the threads drawn straight from them could have been made fast anywhere on the circumference of the tub, provided only that the threads were separate and parallel, and that they were in a horizontal plane, so as to be prepared to act as a left-hand couple. It is important that the student see that these conditions are sufficient as well as necessary.

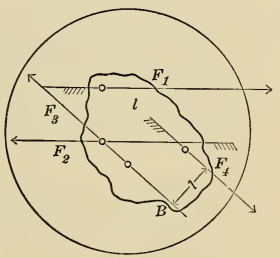


Fig. 8

1. Suppose the body B, Fig. 8, to be acted upon by the two forces F_1 and F_2 forming a right-hand couple with the arm l. It is evident that the "point of application" of F can be anywhere on the line of action, in each case.

2. The direction of the two lines of force can be any direction in the horizontal plane of the couple. For suppose new points of attachment are taken and a new direction to the parallel lines be chosen

as shown by F_3 and F_4, so that both force and arm remain numerically the same (leaving the couple still right-handed) the moment must still be $M = Fl$.

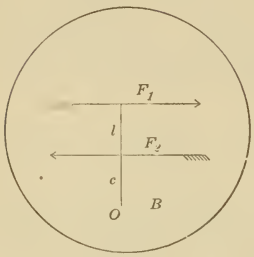


Fig. 9

3. The turning effect with reference to *any point* in the plane of the couple taken as an axis is always the same. Let a vertical at O Fig. 9 be taken as an axis about which the couple M (*i. e.*, the two forces with the arm l) has a tendency to turn the body B. The tendency of F_1 taken alone is evidently measured by $+F(c+l)$, and the tendency of F_2 is obviously $-Fc$. The tendency of the couple is then the sum of the tendencies of its parts, viz.:

$$F(c+l) - Fc = Fl.$$

The arbitrary distance c has disappeared from the equation and there is nothing in the result to indicate where O was taken. This means that whether c be long or short, positive or negative, the result is the same.

21. Hence we conclude, that *a couple acting upon a rigid body can be shifted at will in its own plane to new points and directions without affecting its turning tendency provided only that its moment and sign be unchanged.*

Since a given couple can be balanced by one of any number of left-hand couples constituted as above described, these left-hand couples must have the same moment and be equivalent to each other.

22. We therefore conclude in general that *two couples are equivalent which have the same moment, which lie in the same plane and have the same sign* (that is they tend to turn a body in the same direction). It follows that a couple F_1l_1 may be replaced by a couple F_2l_2 provided $F_1l_1 = F_2l_2$, as: (10 lb) × (6 feet) = (4 lb) × (15 feet); = (20 lb) × (3 feet).

23. The moment of a force. If the axis be taken in the line of action of one of the forces of a couple, the moment of the couple is identical with the product of the other force by its distance from the axis.

Definition. The *product of a force, by its distance from a given point (or axis) is called the Moment of that force with reference to that point or axis.*

24. The turning effect or tendency of a couple is not changed by a shifting of it to a parallel plane.

Proof. Suppose we have two couples M_1 and M_2 equal in all respects, but acting in parallel planes H and K. Suppose in the plane H, a

third couple M_3 equal to M_1 in all respects except the direction of its turning tendency; it may be so shifted as to completely neutralize M_1 and hence balance it. It will now be shown that it can equally well balance M_2 in the parallel plane K. Let M_3 be so shifted that the arms of M_3 and M_2 are parallel, when M_3 becomes M_3' as shown in Fig. 10. It is now easily seen that the four equal forces F_2, F_3, F_2', F_3' balance, for, by symmetry, the concentration of F_2 and F_3 is a parallel force of $2\ F$ thru O, which is midway between them. In like manner the concentration of F_2' and F_3' gives a parallel force of $2\ F'$ thru O, in the direction opposite to $2\ F$, so that the resultant is zero. Hence M_2 and M_3 balance. Now since M_1 and M_2 can separately be balanced by the same couple, they must be equivalent.

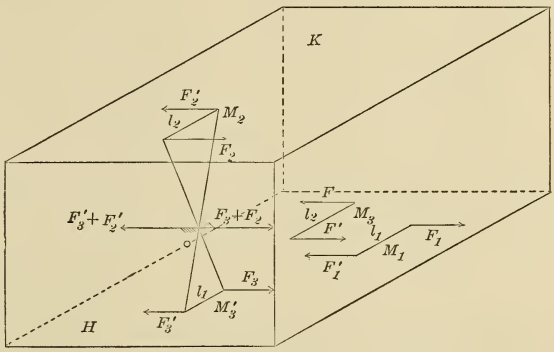


Fig. 10

25. Definitions of the "axis of a couple" and of positive and negative couples.

It has been seen that the point O (see Fig. 9), assumed for the purpose of computing the resulting tendency of a couple, may be anywhere in the plane of the couple. Now through O imagine a line drawn perpendicular to the plane of the *couple*; it will be perpendicular to all the parallel planes to which the couple may be shifted. Hence O may be any point in space, and the perpendicular line will be called the "*axis of the couple.*"

1. This axis is a purely ideal matter; it has no dynamic function. In Statics since the forces which act on a body balance, whether they form couples or not, the *axis* of a couple is not a restraining device, for it does not react upon the body in any way. It is merely an aid in our analysis.

2. Now as to the sign of a couple. When we stand in front of the dial of a tower clock, the hands appear to us to move "right-handed," or "clock-wise." To a person *in the tower*, however, the motion of the same hands appears to be left-handed. It is therefore necessary to determine in every case what "right-handed" means. To do this, take a point O as an origin on the assumed axis, and let one end of the axis, $+X$, be called positive, and the other, $-X$, be negative; and let

$+X$ be so taken that the direction *from* $+X$ *to* O shall be *the absolute direction in which one must look* in order that the couple may appear to him to be right-handed or positive. Left-handed couples are negative. In the case of the Y-axis, look from $+Y$ towards O.

26. The addition of couples having the same axis. Suppose two couples in parallel planes act on the same body. First, let one or both be shifted till they act in the same plane. Second, while preserving the *product* value of *force* by *arm*, let one or both factors be so modified that the forces in the two couples shall have a common magnitude. Third, let their positions be so changed that a force of one couple shall directly balance and neutralize a force of the other couple. Fig. 11*a* shows the two couples in the same plane, and Fig. 11*b* shows them so shifted and modified that two forces neutralize each other, and only one couple remains which has an arm equal to the sum of the two arms, and a moment equal to the sum of the given moments.

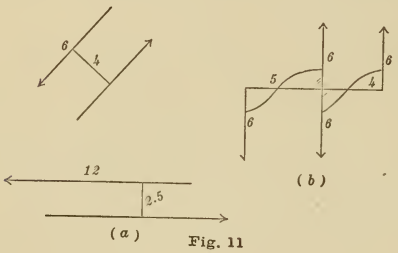


Fig. 11

Example—Let M_1 have a force of 4 (units) and an arm of 5 (units); and M_2 have a force of 6 (units) and an arm of 3 (units).

Let them be shifted and modified while still equivalent, and then combined into one couple $M_3 = 38$.

1. It is evident that any number of couples with a common axis could be added in the same way. A negative or left-handed couple would have the effect of shortening the arm of a positive couple. Show this.

2. An equally simple method of adding co-axial couples would be to reduce them all to a common arm, so that when superposed the result would be one arm with equal and opposite sets of co-linear forces, making a single couple.

Example.—Given two co-axial couples:

$$\left.\begin{matrix} F = 7 \\ l = 8 \end{matrix}\right\} \textit{positive}, \text{ and } \left.\begin{matrix} F = 12 \\ l = 2 \end{matrix}\right\} \textit{negative}.$$

Reduce to a common arm $l = 4$, and combine by superposition.

26[1]. The resolution of couples. The converse of the last section shows at once that a couple may be resolved (or separated) into any number of couples by inserting at different points on the arm equal and directly opposite forces, each of the given magnitude F, and each

parallel to the given forces. The forces introduced balance each other in every pair, and therefore they do not affect the condition of the body acted upon. The given couple $(F)\times(AB)$, is resolved into five component couples, having the same axis, forces of a common magnitude, and different arms.

Still further, every one of the ten forces can be resolved into any number of components (the same for each of the two forces of a couple). Thus each of the above five couples may be resolved into several couples with equal arms, as is the one shown in (*b*).

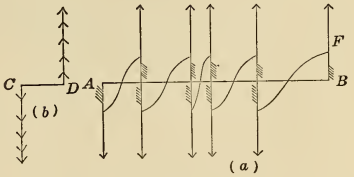


Fig. 12

It thus appears that the original couple may be resolved into any number of component couples.

When these two methods of resolving a couple into components are combined, it is evident that two couples, acting on the same body, having the same axis, and equal moments, may be shown to be equivalent, because they can be resolved into the *same number of equal component couples.*

It follows that a couple which has an arm 7 units long and a force equal to 10 units has a turning tendency 70 times as great as the moment of a couple which has a unit arm and a unit force. Hence the turning tendency or moment is measured by the *product of its force, by its arm, i. e.*, the product of the two numerical measures. This is the proof which was promised in **18.**

The student must never confuse his thought by saying that we can *really* multiply a force by an arm; that would be absurd. What we *really* do is to multiply one *number* by *another number*, and the product is still a number.

What are the "*conditions*" under which it is proper to say that a length of three feet and a force of seven lbs, give us a moment of 21 foot-lbs.?

From all this it follows that the essential things about any couple are: the *direction* of its *axis*, its *moment*, and its *sign.*

27. Couples as vectors. It is now evident that the *essential* things of a couple may be represented by a single straight line or vector: It must have *length* to represent the *moment* of the couple; it must have *direction* to show the axis, and therefore the *plane* of the couple; it must show the "sense" or tendency by **pointing** in the **direction in which the couple must be viewed** to appear right-handed. Couple vectors will be used in discussions which follow.

28. The combination of couples which do not have the same axis, *i. e.*, which are *not* in the *same* or *parallel planes.*

1. Let there be two couples in planes perpendicular to each other, and acting on the same body. They can be combined as follows: Reduce the couples to equivalent couples so that the four forces will have a common magnitude. Next, shift and if necessary *turn* the *couples*, each in its own plane, till there are two equal and directly opposite forces *acting along* the *line of intersection* of the two planes. These two co-linear forces balance each other, and there remain two equal forces which form a couple in a *third plane* which is parallel to the intersection of the first two. The arm of this *resultant couple* is evidently the hypotenuse of a triangle whose legs are the reduced arms of the given couples.

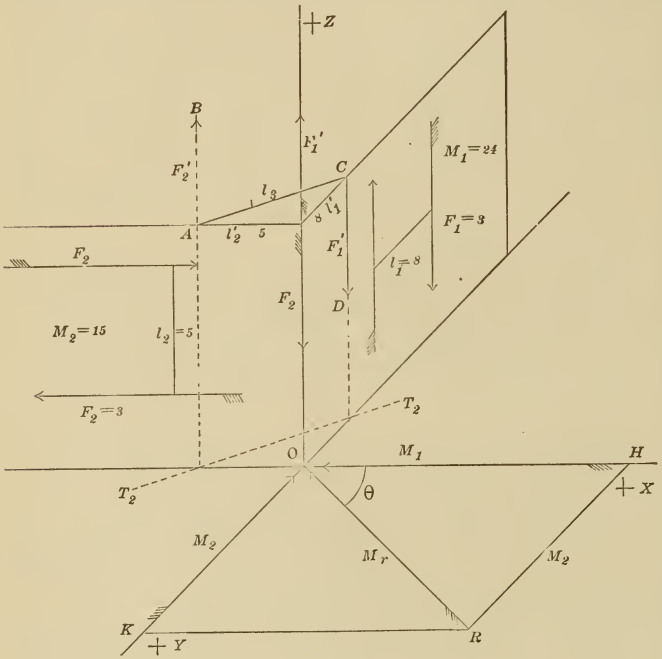


Fig. 13

Illustration. Let OX, OY, OZ be rectangular axes. Fig. 13. Let $M_1=24$ be a couple in the plane YZ, right handed to one *looking* from $+X$ towards O. Let $M_2=15$ be a couple in the ZX plane, right handed to one looking from $+Y$ towards O. Reduce each couple to an equivalent couple with a force 3, and a proper arm; so that $M_1=3\times 8$, and $M_2=3\times 5$. When the reduced couples are shifted to the intersection OZ, so that two of the forces neutralize each other, there remains only a couple having one force in the plane YZ acting down, and the other in the plane ZX acting up, with an arm which equals $\sqrt{(64+25)}$; so that the *resultant couple* is found to be:

$$3\sqrt{(64+25)}=\sqrt{[(24)^2+(15)^2]}=M=\sqrt{(M_1^2+M_2^2)}$$

The axis of M is evidently a line in XY (or parallel to that plane) perpendicular to the trace T_2T_2 of the third plane $ABCD$, on XY.

Fig. 13 also contains a graphical solution of the problem by the method of vectors.

The *direction* of M is determined by the equations:

$$\sin\theta = \frac{M_2}{M}, \qquad \cos\theta = \frac{M_1}{M}, \qquad \tan\theta = \frac{M_2}{M_1},$$

The moment M_1 is laid off to scale on $+X = HO = 24$. In like manner M_2 is represented fully by $KO = 15$, and then M by the diagonal of the rectangle.

3. The student should prove by geometry that the triangles RHO and JIO are similar and that consequently RO is perpendicular to T_2T_2.

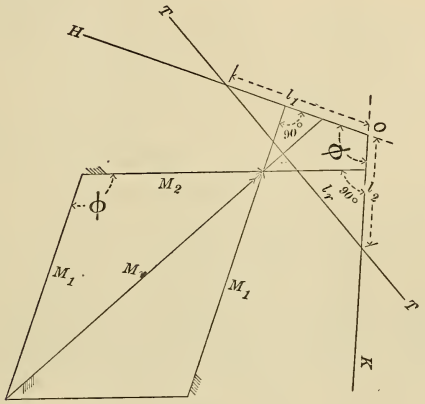


Fig. 14

29. When the planes are oblique to each other the analysis is the same, but the triangle of arms is oblique. 1. The algebraic solution is as follows:—Let the given couple be reduced to equivalent couples, the magnitude of F being common. Then

$$M_1 = Fl_1, \text{ and } M_2 = Fl_2$$
$$\text{and } M = F\,(l_1^2 + l_2^2 - 2l_1l_2\cos\phi)^{\frac{1}{2}}$$

in which ϕ is the angle HOK (Fig. 14.)

2. In like manner the Graphical solution gives M in magnitude and shows the direction of the axis which is perpendicular to the plane of the new couple. $M = \sqrt{(M_1^2 + M_2^2 - 2M_1M_2\cos\phi)}$.

Let both planes be perpendicular to the paper, and let their traces intersect at O. M_1 fully represents the couple in OH, and M_2 fully represents the couple in OK; and consequently M fully represents the resultant couple in the plane TT.

30. It is readily seen that two couples in non-parallel planes cannot possibly balance each other; and it is as easily seen that two couples in non-parallel planes may be balanced by a third couple, and that the moment of the balancing couple must be equal to M, the resultant of the other two, and the arrow line of its axis must be just the opposite of that of M.*

*A resultant couple has been defined as the couple which balances the couple which balances the given set of couples.

31. Three couples balance. It follows from the last two problems that when the representative vectors of three couples, are of such lengths and directions that they may form a plane triangle, when they are drawn *in order, as they point*, they balance, and the body on which they act is in equilibrium.

Thus the triangle ABC represents fully three couples acting on the same body, each with a definite tendency to turn it in a definite direction, and yet they balance and (so far as these couples are concerned) the body is at rest. The triangle ABC is called the *moment triangle of equilibrium.*

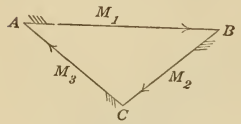


Fig. 15

32. The resultant of three or more couples. Having found the resultant of two couples as shown above, a third couple may be combined with the resultant of the two, and a resultant of three be found. This last may then be combined with a fourth, and so on for any number of couples.

1. When the planes of three couples are the rectangular co-ordinate planes, the problem is simple. The resultant of M_1 and M_2 is $M' = \sqrt{(M_1^2 + M_2^2)}$; and the resultant of M' and M_3 is

$$M = \sqrt{(M_1^2 + M_2^2 + M_3^2)};$$

its vector is the diagonal of a rectangular solid constructed on M_1, M_2, and M_3 as confluent edges.

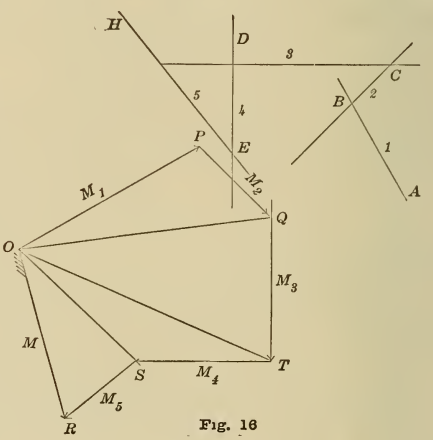


Fig. 16

2. If the planes of several couples are perpendicular to a common plane, the graphical solution is most simple. In Fig. 16 let the plane of the paper be the common perpendicular plane. Let the planes of the given couples have the traces AB, BC, CD, DE, EH, etc. From any point O (Fig. 16) draw M_1 fully representing the couple in AB; from the end of the vector M_1, draw M_2 fully representing the couple in BC; OQ, then, fully represents the resultant of M_1 and M_2. From the extremity of the vector M_2, draw M_3, then M_4, and so on. OT is the resultant of M_1, M_2 and M_3; and $OR = M$ is the resultant of them all. The *balancing couple* is represented by the vector RO, pointing from R towards O.

If the planes of the given couples have no common perpendicular

plane, the polygon $OPQSTR$ is not a plane figure; it is a polygon in space, but OR is still the vector of the resultant couple.

33. The combination of a couple and a single force acting in a plane parallel to the plane of the couple.

Suppose that among the forces which act upon a body at rest there are two which form a couple and a third force whose line of action is in, or parallel to, the plane of the couple. Let the irregular outline (Fig. 17) represent the stationary body; in this case, let the curved arrow and the numerical value of M represent the moment of the couple; and let the force F be represented by OA. It will now be shown that the resultant of M and F is the single force F_r.

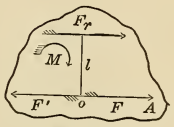


Fig. 17

(*a*) If M and F are not in the same plane, shift M to a parallel plane containing F.

(*b*) In place of the original force and arm of the couple, let $M = Fl$, in which F has the magnitude of the given force F, and $l = M/F$.

(*c*) Shift and turn the couple till one of its forces F', has the line of action of the given force F, but acting *opposite* to it, thereby canceling it, so that nothing is left but the *force F_r acting parallel* to the *given* F, and of the same magnitude. *The distance moved to the left* is $l = M/F$; that is, if an observer standing at O sees the couple right-handed as he looks *down*, the *Resultant* of M and F, will be F_r on his *left*, when he wheels about and looks in the direction OA.

Problem. The student should show that a given *left*-handed couple would have caused the resultant force to appear on his *right*.

34. The converse of the above proposition is evidently true, viz.: A given force F may be resolved into a couple and an equal force F acting at any point in space, distant l from the line of action of F, the moment of the couple being Fl and its axis being perpendicular to the plane of F and l.

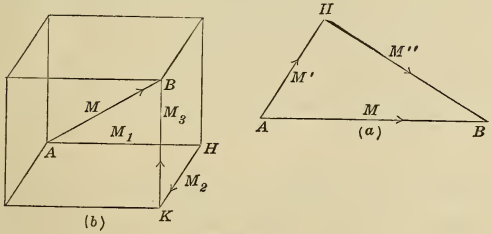


Fig. 18

35. The resolution of couples. A given couple may be resolved into two or more couples graphically, if the vector of the given couple is the line AB (Fig. 18 *b*), by drawing two component vectors AH and HB; or by three vectors which may be the edges of a rectangular solid. (Fig. 18 (*a*)).

36. Resolution of a force into a force and two couples. A single force acting on a body, may be resolved into two couples and an equal single force acting at any point on the given body that may be chosen.

Let the point A (Fig. 19) be the point of action of a given force F. It may be resolved by the introduction of two balancing forces $+F$ and $-F$, at B, into an equal parallel force at B, and a couple Fy, right-handed as seen from $+X$, as shown in the figure. Still more, the force $+F$ at B may be itself resolved into an equal parallel force $+F$ at O, and a couple $(-Fx)$ (which is left-handed seen from $+Y$). We have thus resolved a force, acting at the point (x, y) and parallel to OZ, into an equal force acting at O, and two couples; one having a moment $+Fy$ about the X-axis, and one having a moment $-Fx$ about the Y-axis.

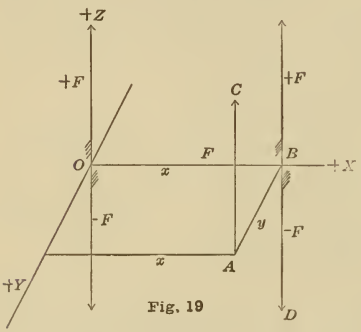


Fig. 19

The use of this method of resolution will be illustrated by several examples in Chapter III, and will be freely employed in subsequent chapters.

CHAPTER III.

Parallel Forces Which Balance.

37. Three parallel forces balance. Since they could not balance, if all acted in the same direction, the largest must act in a direction opposite to that of the other two; hence we must have

$$F_1+F_2-F_3=0$$

or, numerically,

$$F_3=F_1+F_2. \qquad (1)$$

If we resolve F_3 into two forces respectively equal to F_1 and F_2, we have two couples which must balance each other. Since they could not balance if they were not co-axial, and since the couples cannot be co-axial unless the forces are co-planar, we reach one im-

portant conclusion: that **three parallel forces cannot balance unless they act in the same plane.**

Since the couples balance, their moments have equal magnitudes and opposite signs.

Ex. Let the three forces be represented by a scale drawing, Fig. 20. As we look down upon the drawing, F_1b is right-handed, and F_2a is left-handed. Hence $F_1b - F_2a = 0$,

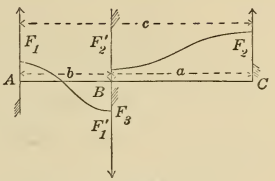


Fig. 20

$$\text{or} \quad \frac{F_1}{F_2} = \frac{a}{b} \tag{2}$$

or the *forces* (of the balancing couples) are inversely as their *arms*.

If we let the distance between the forces having the same direction be known as c, we have in this case

$$c = a + b$$

Now erect an axis at A, and take moments. We have at once

$$F_3b = F_2c, \text{ hence } \frac{F_2}{F_3} = \frac{b}{c} \tag{3}$$

If we take C as our axis, and take moments, we have

$$F_1c = F_3a \text{ hence } \frac{F_1}{F_3} = \frac{a}{c} \tag{4}$$

Combining (2), (3), and (4) we have

$$F_1 : F_2 : F_3 = a : b : c; \tag{5}$$

a very useful *triple proportion*.

38. Deductions. From the above discussion several important conclusions follow. When three parallel forces acting on the same rigid body balance, we know:

1. The three lines of action lie in the same plane.
2. Their algebraic sum is zero.
3. The line of action of the largest force lies between the lines of action of the other two.
4. The magnitude of each force is proportional to the distance between the other two.
5. In the language of algebra

$$\begin{gathered} F_1 + F_2 + F_3 = 0 \\ F_1 : F_2 : F_3 = a : b : c \\ a + b = c \end{gathered} \tag{6}$$

When a body acted upon by three parallel forces is at rest, we know that the above equations must be true.

Problems.

1. Given two parallel forces 18 lb and 40 lb, 8 feet apart. Find the balancing force, and the distances between it and them.

2. Given the parallel forces $P = +80$ lb, $Q = -64$ lb, and the distance between them, 12 feet. Find the balancing force, and its position.

3. Three parallel co-planer forces act on the same straight bar at given points A, B, and C, and balance. The force at $A = 35$ lb. Find the forces at C and B.

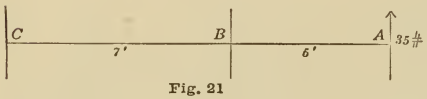


Fig. 21

Use the proportions in equation (6) for magnitudes.

4. Suppose the force A makes an angle of 30° with the bar. How does the obliquity affect the result?

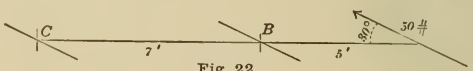


Fig. 22

39. The resultant of two parallel forces. Since F_1, F_2, and F_3 balance, it is evident that one must balance the combined action of the other two. Now the single force which is equivalent to the combined action of two or more forces is called the "Resultant" of those forces. In the present case the resultant of two must be equal and directly opposite to the balancing force*. Hence if we know all about the third or balancing force, we know all about the resultant.

Thus, if P, Q and S, acting on the same body, balance, the resultant of P and Q is directly opposite to S; and the resultant of P and S is directly opposite to Q.

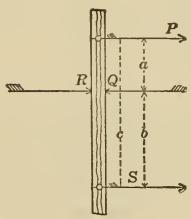


Fig. 23

The student should adapt the equation (6) to this case; and write deductions as to the *resultant* as was done for the balancing force in the last section.

40. Graphical solution for the resultant of two parallel forces. The following elegant solution is taken from Rankine's Applied Mechanics.

Ex. Given, in Fig. 24, two parallel forces, F_1 and F_2, fully represented in magnitude, relative position, and direction by AP and BQ. To find their resultant graphically.

Solution. 1. Draw thru P and B lines parallel to the (imaginary) line AQ. 2. Draw thru A and Q lines parallel to the (imaginary)

* This justifies Rankine's definition: "The resultant of a system of forces is the force which balances the single force which balances the given forces."

line PB. The intersection of the new lines thru P and Q will give the arrow head of the resultant; and the two lines thru A and B will intersect at the tail end of the resultant, which is therefore CR.

In (a) Fig. 24, the given forces act in the same direction: in (b), they act in opposite directions.

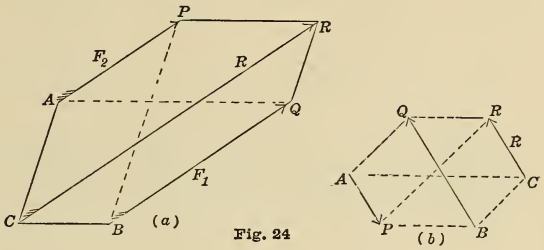


Fig. 24

The geometrical proof of the correctness of these solutions is left to the student. He should prove in each case that CR fulfills all requirements of *magnitude, direction and position.*

41. The lever. Case I. Suppose a rigid beam carrying a hook at each end is balanced on a knife edge, *i. e.*, the sum of the pulls of the earth on the beam and hooks is equivalent to a single pull directly over the knife edge. This pull (or weight) is balanced by the upward lift of the knife edge, or "*Fulcrum*," so that when we come to consider the action of other forces on the beam, we can completely neglect the weight of the beam itself.

Let us represent, Fig. 25, the distance from hook to hook by l, and the length of the two arms by a and b respectively.

Suppose F_1 and F_2 are applied to the hooks and that they balance. We have at once

$$R = F_1 + F_2$$

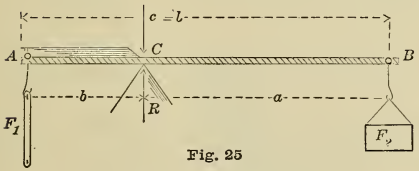


Fig. 25

which must be the additional support given by the fulcrum —to balance F_1 and F_2.

From the proportion equation derived from (6)

$$a : b : l = F_1 : F_2 : R$$

$$a = \frac{F_1 l}{R} = \frac{F_1 l}{F_1 + F_2}, \text{ and } b = \frac{F_2 l}{R} = \frac{F_2 l}{F_1 + F_2}.$$

Again, if F_1, F_2, and a are given, we have as before, $R = F_1 + F_2$, and from the proportion

$$\frac{F_1}{F_2} = \frac{a}{b}, \quad b = \frac{F_2}{F_1} a$$

and $l = a + b$.

If F_1 and the two lengths, a and b, are known

$$\text{then } F_2 = \frac{b}{a} F_1$$

$$\text{and } R = \frac{a+b}{a} F_1.$$

This case was formerly spoken of as a "lever of the **first class.**"

42. Case 2. Let the lever AB and its hooks be balanced on C by means of a counter weight D, so that the weight of lever and counter-weight may again be neglected. See Fig. 26. Let F_2 be a known force acting down as shown at a distance a from C. An unknown force acts up at B whose distance from F_2 is c. We are to find the force F_1 and the additional load put upon the support at C.

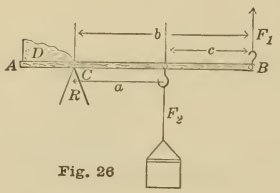


Fig. 26

It is evident at a glance that the turning moment of F_1 about a horizontal axis at C is negative (left-handed), and equal to $F_1(a+c)$. It is also evident that the turning moment of F_2 about the same axis is right-handed, and equal to $F_2 a$.

These two moments must balance, hence

$$F_1(a+c) = F_2 a$$

so that

$$F_1 = \frac{a}{a+c} F_2.$$

But $a+c=b$ as defined for the lever, so that we have

$$\frac{F_1}{F_2} = \frac{a}{b} \text{ as in formula (6).}$$

The resultant of F_1 and F_2 must be equal to their difference and it must act opposite to the support of the fulcrum, as the latter is the balancing force. Hence the fulcrum action due to F_1 and F_2 is

$$R = F_2 - F_1$$

Fig. 26 illustrates what was once known as a "lever of the **second class.**"

43. Case III. **A lever of the third class** is illustrated by Fig. 27.

Ex. 1. A man, with a pitchfork six feet long, supports at the far end a mass of hay weighing 24 lb. One hand is at A, acting down, and the other hand at B, 18 inches away, is acting up. What are the

forces at A and B if the handle is horizontal and all actions are vertical?

The hand at A may be thought of as the fulcrum.

Ex. 2. In the above example, for pitchfork substitute fish pole; for 6 feet, put 20 feet, and for hayweighing 24 lb, put a fish weighing 5 lb. Find the forces at A and B, capable of just supporting the weight of the fish.

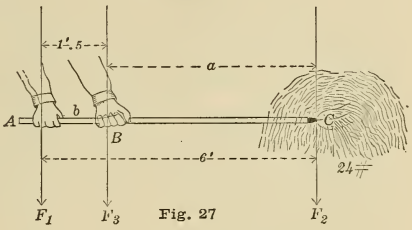


Fig. 27

44. A Word about Friction. Whenever the surface of one body is in contact with the surface of another, there is always a more or less effective interlocking of the minute particles of one body with the particles of the other, so that they are able to resist separation by sliding. This ability to resist sliding (or relative motion) is variously called *adhesion*, *cohesion*, *sticking*, or *friction*. The term *Friction* is used in Mechanics as the name of the force with which the bodies in contact resist a mutual sliding.

There is, of course, a limit to the ability of material in contact to resist a sliding motion. When the external forces tending to produce sliding are so great that the limit of frictional resistance is passed, then sliding takes place.

Fig. 28 represents a "rough" heavy body resting upon a horizontal plane which is not smooth. The body is stationary in spite of the tension T in a rope which *tends* to draw it sideways. As the body A stands still, the slender blocks (b) stand on end in equilibrium. The friction between A and B is sufficient to balance T. Between A and B there are two different actions both distributed over the surface of contact, one *normal* or vertical, one *tangential* or horizontal. Each of these forces is represented by two arrows. The two arrows on A show how A is acting on B; the two arrows on B show how B is acting on A. It is evident that B's horizontal action on A is equal to the horizontal action of the rope T, and in the opposite direction; hence they balance.

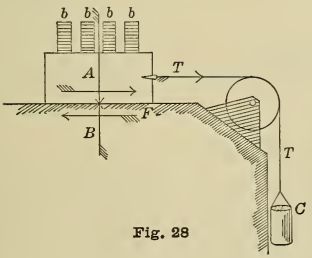


Fig. 28

Let F denote the friction shown by the horizontal arrow on A or B. Since A stands still we have

$$F = T$$

If more shot or pebbles are put into C, T will be increased, and of course F will be increased also, *as long as A stands still.* When T increases to a certain point, A starts and slides. F has reached its limiting value F_1, and the *difference* between T and F_1 makes A move and perhaps causes the slender blocks to fall over.

In statical problems the limit of frictional ability is not reached. The laws which govern friction will be discussed later when motion is studied, but it is well, at this point, to remark that the frictional ability of surfaces in contact depends upon several things: the normal action (pressure) between the two; the character of the surfaces (rough, polished, or lubricated), the temperature, etc. In Mechanics, the word "smooth" is used only in cases of ideal bodies where the friction is zero.

45. Applications. CASE I. Show how the action of steam on the piston of a locomotive engine has a tendency to move the loco-

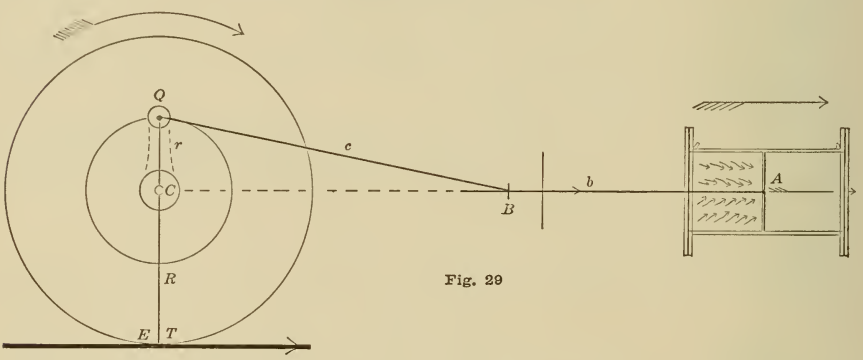


Fig. 29

motive; *first, when the crank pin is above the axle* of the driving wheel (Fig. 29).

Solution. 1. Suppose the steam at high pressure is as shown in the cylinder. It presses on the piston towards the right and equally on the cylinder head towards the left. The cylinder is bolted to the frame, so the action of the steam on cylinder head is to push or move the frame of the engine *to the left.* If the area of the piston face is A square inches, and the steam pressure per square inch is p lbs., in excess of the "*back pressure*" on the other face, the force tending to *move the frame to the left,* or backwards, is Ap.

The steam pressure, Ap, upon the piston is transmitted thru the piston rod b to the crosshead B, thence thru the connecting rod c to the crank-pin Q. Omitting for the present the small amount of frictional resistance offered:—by the cylinder upon the circumference

of the piston, by the packing in the cylinder-head upon the piston-rod; and by an upper guide D upon the cross-head,—it is clear that the horizontal force acting *to the right* upon the crank-pin at Q is Ap.

The crank and wheel form a *rigid lever* which (omitting the pull of the earth which is balanced by the vertical action of the track) is acted upon by *three* forces which balance. (See Fig. 30.)

First, The force Ap on the crank pin;

Second, The *frictional force* of the track upon the circumference of the wheel at E, which force we will call T. This force prevents slipping.

Third, The *reaction* of the frame thru the journal box at C upon the axle, which reaction must equal the sum $Ap+T$. Thus we see that the *direct action* of the *axle upon the frame* is the *resultant* of Ap and T, which is $Ap+T$ *acting towards the right.* We found in the first paragraph of this solution that the action of the steam upon the cylinder-head tended to move the engine frame of the locomotive to the *left,* by a force of Ap pounds. We have in the last sentence found that the action of the axle of the driving wheel tended to move the frame to the *right* by a force of $Ap+T$ pounds. The algebraic sum of these horizontal forces is

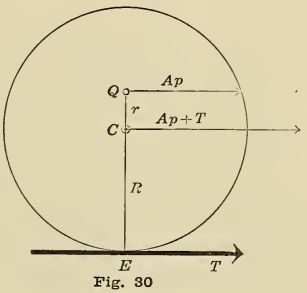


Fig. 30

$$Ap+T-Ap=T$$

which is exactly the horizontal action or *friction* between the track and the wheel.

We shall find the numerical value of T by solving the problem of the lever consisting of the radius R and the crank r. By the law of the lever

$$\frac{Ap}{T}=\frac{R}{r} \text{ or } T=\frac{rAp}{R}$$

which is the measure of the resultant forward action of the steam in our cylinder upon the locomotive; in other words, this is the force which tends to move the locomotive, and which *will* move it unless some force not considered above holds it. The student must not fail to see clearly that the action T of the *track upon the wheel* is *forward,* and hence the locomotive moves forward.

46. Second, when the crank is below the axle. Case II. See Fig. 31.

1. In this case the action of the steam upon the *cylinder head* and thru it upon the frame of the locomotive, is *Ap to the right.*

2. The action of the steam upon the *piston* puts compression or thrust into the piston-rod and into the connecting-rod, and has a *tendency* to move the *crank-pin towards the left.*

3. As before, the rigid body of wheel and crank forms a lever,

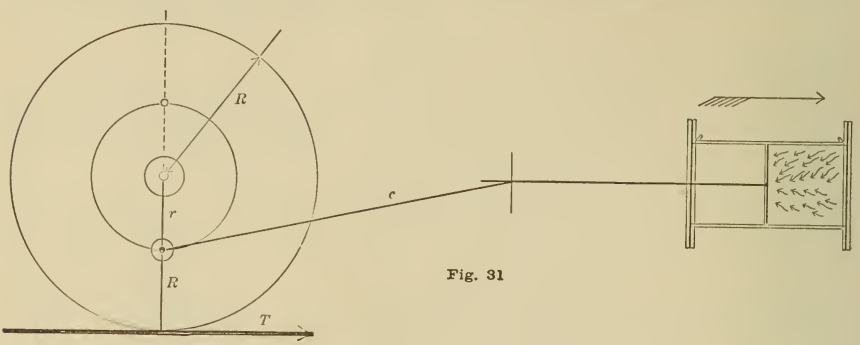


Fig. 31

with two parallel forces, Ap and T *acting in opposite directions.* Fig. 32. Their resultant acts *towards the left* thru the axle upon the frame, namely, $Ap - T$. (See the lever of the third class), **42.**

4. The sum of the actions upon the frame is Ap on the cylinder-head, which is *positive* to *the right*, and $Ap - T$ which is *negative* to the *left.* Their resultant Action is:

$$Ap - (Ap - T) = T,$$

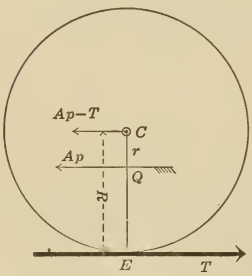


Fig. 32

and T is *just the same as before.* This shows that the steam is urging the locomotive forward *just as much* when the piston rod is pushing back, as when it is pulling forward.

In the foregoing discussion it is very important to keep in mind the vast difference between the action of the *Frame upon* the driving *axle* (thereby making a balancing force with T and Ap), and the action of the *axle upon the frame* thereby being the resultant of T and Ap. It is this *resultant* which counts in the discussion.

47. Remarks. From the value of $T = \frac{rAp}{R}$ in each case, it appears:—

1. That the higher the steam pressure p, the greater the traction.
2. The larger the cylinder cross-section, the greater the traction.
3. The longer the crank-arm, the greater the traction.
4. The smaller the radius of the wheel, the greater the traction.
5. Since T is limited by the frictional ability of the surfaces in contact (wheel and rail) which ability depends, as everybody knows,

upon the load on the wheel, and the character of the surfaces (clean or sanded, wet or dry, cold or hot, greasy or icy, etc.) there is in every case a greatest possible value of T, which we will call T_1.

Since the value of T_1 is made greater by increasing the roughness or "biting" ability of the rail, the engineer, when he needs greater traction, operates a device which sprinkles sand upon the rail. But even then there is a limit to the value of T_1.

Hence the greatest value of the steam pressure is

$$p_1 = \frac{RT_1}{Ar}.$$

If the engineer attempts to increase T_1 by increasing p_1, the wheel will *slip*, and as is well known the traction at once falls below T_1. However, the traction can be increased by the addition of one or two driving wheels, each a duplicate of the first and each carrying, approximately, the same load. The crank or cranks must be similarly placed, and the new connecting-rod, or rods, from wheel to wheel, must always be horizontal. The action of the main connecting rod is then distributed between the wheels, thus permitting a greater steam pressure without causing the wheels to slip. T is then ΣT.

Problem.

Let the student make skeleton drawings of a horizontal engine driving a paddle-wheel steamboat, and then write out analyses similar to the above.*

The problem in the text is introduced to show that the principles of the lever enable one to explain how steam drives a locomotive. The formula found holds for a *single* cylinder and *only when the crank pin is directly above* or *below the axle.*

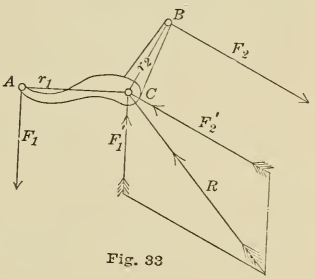


Fig. 33

46. Bent Levers. Bent levers are analyzed as are straight levers by the use of moments. A lever (without weight) acted upon by forces F_1 and F_2, with lever arms r_1 and r_2 respectively, is in equilibrium if the moments F_1r_1 and F_2r_2 are equal, *provided* the axle at C is properly supported. The angle ACB may have any value.

* The captain of a Mississippi steamboat once tried to convince the author that his engine propelled the boat only when it was pulling forward with the crank pin above the shaft. When the crank was below the axle and the steam was pushing back, he thought the engine was a hindrance and not a help. He knew so little of the principles of Applied Mechanics that no reasoning could convince him that he was wrong; but his boat went ahead with *every stroke* just the same.

Hence $\frac{F_1}{F_2} = \frac{r_2}{r_1}$ or the forces are inversely as the arms, as in the case of a straight lever.

The Support reactions. The reactions upon the fixed axis of a bent lever must not be overlooked. In the last figure the force F_1 calls into being the equal acting force $F_1' = F_1$; and the force F_2 in like manner brings into play the action $F_2' = F_2$. (See **19.**) The resultant of F_1' and F_2' (see Chapter V.) is the force

$$R = \sqrt{(F_1^2 + F_2^2 - 2F_1F_2\cos ACB)},$$

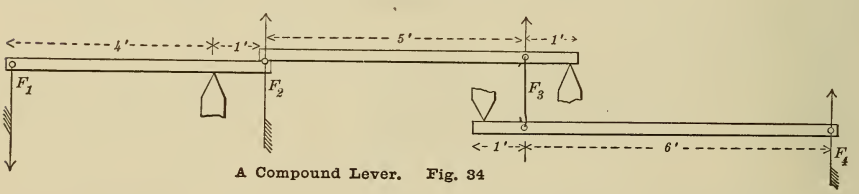
which *must be balanced* by the *reaction* of the journal box of the axle. The magnitude and direction of this supporting force must never be ignored.

47. Compound levers. The chief use of levers is to transmit and modify force. A force is applied to a lever at A, and the result is a very different force applied to a body at B. A very small force at A may produce a very large force at B, given a strong lever and a sufficient support as a fulcrum, and there is no limit to the ratio of the force at B to that at A. Hence, the first great Engineer of history, Archimedes, said:—"Give me a spot whereon to rest my lever [that is, a fulcrum] and I will move the world."

1. It is often necessary or at least profitable to separate a ratio into the product of two or more ratios, which means that one lever may be made to operate a second lever, and, if needed, the second may operate a third. Thus: Fig. 34.

It is evident that

$$F_2 = 4F_1;\ F_3 = 6F_2 = 24F_1;\ F_4 = \frac{F_3}{7} = \frac{24}{7}F_1.$$



A Compound Lever. Fig. 34

The first lever is of the "First Class"; the next of the "Second Class," and the last is of the "Third Class." In every analysis the fulcrum should be taken as the moment axis. A compound lever may be designed to make the ratio F_4 large or small. In Fig. 34 the levers must be so proportioned as to balance when F_1 is zero, so that their weight may be neglected without error. The same will be true of those examples which follow unless otherwise accounted for.

2. Fig. 35 is equivalent to a compound lever consisting of three simple levers. The student will readily see that, assuming no friction and no stiffness in any rope (a very violent assumption):

$$\frac{W}{F_1} = \frac{R_1}{r_1} \cdot \frac{R_2}{r_2} \cdot \frac{R_3}{r_3}$$

That is, the action of W is exactly balanced by the action of F_1, thru the intervening cords and pulleys. If there be friction at the axles, or stiffness in the cords, then the weight W will remain in equilibrium in spite of some departure from the value given by the formula:

$$F_1 = \frac{r_1\ r_2\ r_3}{R_1 R_2 R_3} W.$$

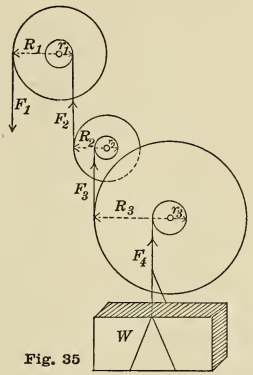


Fig. 35

In Fig. 35, all pulleys and drums are mounted on fixed supports. In the same way gear wheels with their axles act as levers in a train of wheel work. Combinations of pulleys with fixed and with movable blocks will be discussed later in the book.

Problem.

48. Pullman Car Truck. In an ordinary Pullman car, one-fourth of the weight of the car body and contents is brought to each set of three wheels on one side of a truck. It is then by a system of levers distributed in such a way that each wheel carries the same load, viz.: one-twelfth of the entire weight of the car.* The student should inspect a truck, and by means of a drawing explain how this is done.

Testing Machine of St. Louis Bridge.

49. A fine example of compound levers is found in the testing machine designed by Col. Henry Flad and used by Capt. James B. Eads in testing the strength and elasticity of the materials which were put into the great Eads Bridge at St. Louis. The arrangement of the levers is shown in the "figured" drawing, Fig. 36.

This machine was capable of exerting a tensile or compressive force of 100 tons, by means of fluid pressure in a thick iron cylinder, and a system of compound levers.

*A car "truck" is the low steel carriage with six wheels upon which one end of the car-body rests.

When a specimen of steel was to be tested in tension it was coupled in between the two pins shown in the horizontal section near the figure (6″) in Fig. 3. The draw-bar, or piston-rod, is of steel 3 inches in diameter; it passes into the cylinder and is screwed into the plunger or piston. Liquid, glycerine and water, was forced into the cylinder by a force pump operated by hand. The plunger was $7\frac{1}{4}$ inches in diameter, and a liquid pressure of 6,000 lbs. per square inch gave a total of 100 tons. The cylinder itself was held in place by large lugs between blocks on the frame of the machine.

When a specimen was to be tested in *compression* it was inserted between the parallel steel faces on either side of the letter *E*. When the plunger was forced out it pushed the specimen against the outer cross-head (on the left) which brought the four long tension rods into action; they transferred their joint action to the short arm of the first lever, as was done directly in the former case.

The extreme delicacy of the apparatus is shown by the fact, reported by Capt. Eads, that when a specimen was under a stress of nearly 100 tons, a common lead pencil dropt into the can at *K* would cause the long arm of the last lever to fall; and upon the removal of the pencil the arm would rise again. Every lever turns on a steel "knife-edge" which acts as a fulcrum.

The first lever is a bent lever of the first class. Its short arm is $10\frac{1}{4}$ inches; its long arm is 9 feet $4\frac{3}{4}$ inches; so that the ratio is 1 to 11.

The second lever is of the third class. The short arm is 12 inches; and the long arm is 13 feet. Hence the ratio is 1 to 13.

The third lever is a simple straight lever of the first class. Its short arm is 7 inches, and the long arm is 8 feet 1.86 inches, so that the ratio is 1 to 13.98.

All the levers were balanced so that the third lever, which should oscillate between very narrow limits, stood horizontally when there was neither tension nor compression in the specimen. A suspended pan shown at the end of the third lever was in great part the balancing force for all levers.

When the liquid used to drive the piston or plunger in the thick cylinder was applied, the action began upon the specimen, which had been inserted into the machine. The balance of the levers was maintained by pouring fine shot into the pan, and the weight *W*, of the shot, having been carefully weighed in an accurate balance (such as is used in a chemical laboratory) furnished the means for measuring the tensile and compressive stress applied to the specimen. The formula for such a calculation was:

If *P* denotes the required total stress

$$P = 11 \times 13 \times 13.98 \times w = 1999.2\ w = 2000\ w, \text{ very nearly.}$$

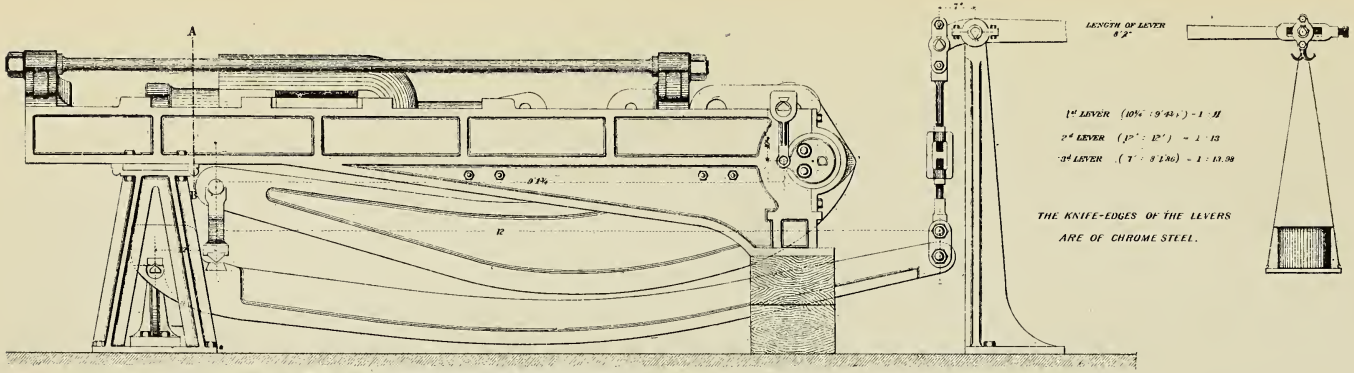


FIG. 2. LONGITUDINAL SECTION CD.

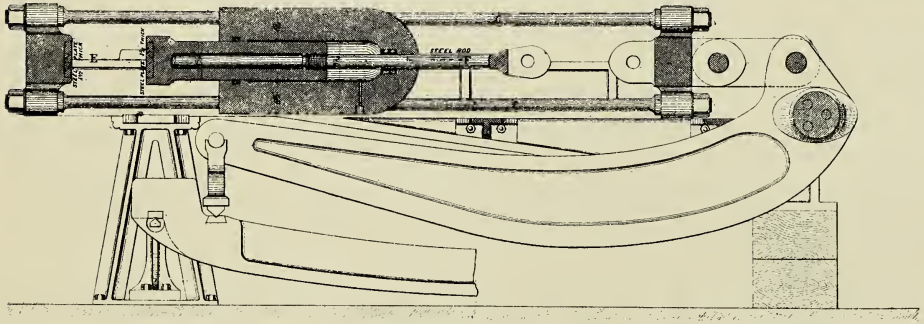
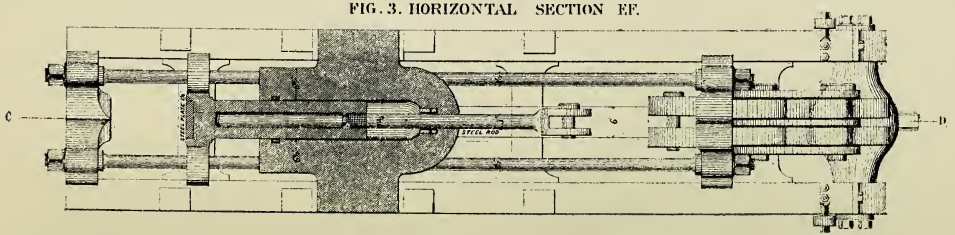


FIG. 3. HORIZONTAL SECTION EF.



ST. LOUIS TESTING MACHINE.

Fig. 36

If w was 100 pounds, P was 100 tons, which was regarded as all that it was prudent to apply to the machine.*

Suppose a steel cylinder 8/10 of an inch in diameter is under tension, and that the shot dropped into the can, to produce a balance, weighs 8 lbs. and $3\frac{1}{2}$ ounces. What is the tension *per square inch* of the cross-section in the specimen?

47. Another warning. The student who has seen, in what has been shown, how a small force at one point can by means of levers produce or balance a large force at another point, must not jump to the conclusion (as so many self-styled inventors have done) that the increase in the force thus produced is all clear gain. Levers are of practical use, as abundant experience shows, but they do not enable one to get something out of nothing.

Levers are generally used to move things, to lift heavy loads, to do work. We shall discuss *motion* later on, but it is timely to say now, that when motion results, there is full compensation for any apparent gain in force by a loss of motion; and a balancing loss in force for any gain in distance moved.

Ex. For example, if a force of 20 lbs. (see Fig. 37) can balance a weight of 1,000 lbs., which is 50 times as great, the body pulling down on the rope at P must pull the rope down 50 feet in order to raise W one foot.

See chapter XXI on Work and Energy.

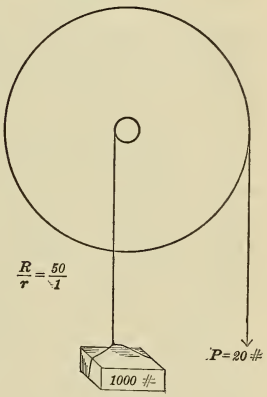


Fig. 37

48. The Relation between any number of Parallel Co-Planar Forces acting upon a body in equilibrium. Since the forces balance, there can be no *resultant* tendency to move the body in any direction or to turn it around. Hence, the algebraic sum of the forces must be zero; and the sum of their moments with reference to an axis erected at *any point* in their plane must also be zero. Hence, the *Equations* of *Equilibrium*.

$$\left.\begin{aligned}\Sigma F &= 0\\ \Sigma M &= 0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

* This elegant machine is still in use in a St. Louis Manufacturing establishment. Figure 36 is reduced from a lithographic plate in the author's "History of the St. Louis Bridge."

For further description of the machine, and the elegant method employed for determining elongation, modulus of Elasticity, and Elastic limit, devised by Chancellor Chauvenet of Washington University, the reader is referred to the History, published in 1881.

In point of fact the balancing forces can always be resolved into a number of separate couples which balance.

1. Example. Let the forces be $F_1 \ldots\ldots F_4$ in Fig. 38. The force $F_3 = 16$ can be resolved into $8+4+4$, which, taken with $F_1 = 8$, $F_2 = 4$, and $F_4 = 4$, form three couples whose algebraic sum is evidently zero, just as $\Sigma M = 0$ for any axis as O.

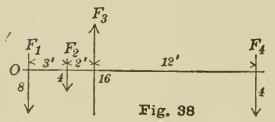


Fig. 38

2. Example. Six parallel co-planar forces act upon the bar AB, as shown in Fig. 39. What and where is the single force which will balance them? Take the line AB as the axis of X, and any point on it as O, as an origin. Let the unknown force be P and its distance *to the right of* O be X. Substitute in the Eq's of Equilibrium, and solve. Let forces acting down be positive so that $(+F)\ (+x) = +Fx$ a right-hand moment.

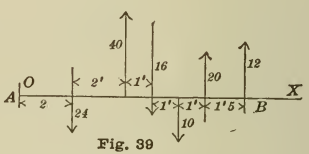


Fig. 39

3. Ex. Suppose we have a rigid frame $ABCD$, lying on a smooth horizontal plane (or floating on the horizontal surface of water), so that its weight is balanced and left out of account. Suppose further that the frame is acted upon by a set of parallel forces all in a horizontal plane, and that they balance. To make the situation seem concrete, let every arrow in Fig. 21 represent the tension in a wire which is fastened to a pin in the frame. Every wire is connected with a spring balance, which does not appear in the drawing. All dimensions are known, and all the forces are given with the exception of two: P and Q. These two are to be found.

Solution. Since the forces balance, there can be no resultant force in any direction, and no resultant moment. Hence we must have $\Sigma F = 0$ and $\Sigma M = 0$; in which summations both P and Q are included.

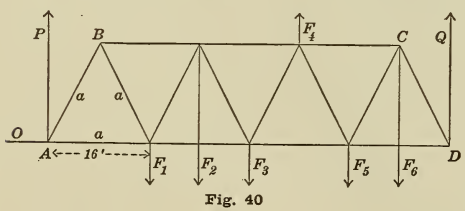


Fig. 40

If the moment axis is taken in the line of action of one of the unknown forces, that force will not appear in the moment equation, since a force cannot have a moment about an axis which it intersects. Hence, at first take the axis at A.

Let the triangles of the frames be equilateral with a side 16 feet long.

Let $F_1 = 350$ lb $F_3 = 100$ lb $F_5 = 200$ lb
$F_2 = 600$ lb $F_4 = -740$ lb $F_6 = 400$ lb

The equations of equilibrium then become

$$350+600+100-740+400-P-Q=0$$
$$\text{or } P+Q=910; \text{ and}$$

$$16\times350+24\times600+32\times100-40\times740+48\times200+56\times400-64Q=0.$$

It is at once evident that this last equation is clumsy; a "half-panel" length could have been used as a unit of length instead of the foot, so that the "arms" would have been 2, 3, 4, 6, etc. Thus:

$$2\times350+3\times600+4\times100-5\times740+6\times200+7\times400-8Q=0$$
$$\text{or } Q=400\text{lb}$$
$$\text{and therefore } P=510\text{lb}$$

Let the student take his axis at D (thereby eliminating Q) and recalculate the value of P.

The student must not suppose that the equation $\Sigma(F)=0$, and *two* moment equations, arising from the use of two different axes, furnish three independent equations sufficient for finding three unknown forces. In fact, the moment equations are not independent.

49. Again, suppose the frame is a bridge truss (Fig. 41) in a vertical plane carrying half of a loaded roadway. Let it be assumed that all forces are vertical and coplanar, and that the loads are placed at the joints. Assume any convenient dimension for the sides of the triangles which are equilateral. Find the supports at A and B. It will be well to let every bar in the frame have the length 2. It will be seen that the *absolute* length has no effect upon the values of V_1 and V_2.

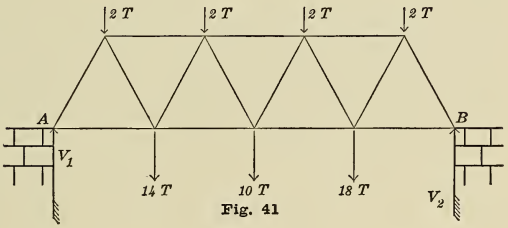


Fig. 41

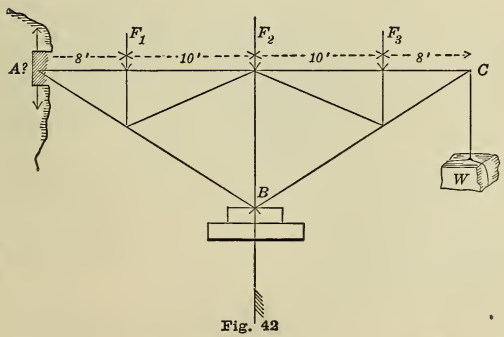


Fig. 42

Ex. 1. Figure ABC (Fig. 42) is a cantilever truss, designed to support a heavy load at C; and smaller loads at the upper pins. It is assumed to be rigid in a vertical plane.

Let $F_1 = 800$ lb $F_3 = 500$ lb $F_2 = 1{,}000$ lb $W = 1{,}600$ lb

Find the supporting forces at A and B.

Ex. 2. Let $F_1 = 1{,}200$ lb $F_3 = 200$ lb
$F_2 = 1{,}000$ lb $W = 60$ lb

Find the supporting forces at A and B.

50. To find the resultant of a set of parallel co-planer forces which do not balance. The *magnitude* of R is evidently equal to the algebraic sum of the given forces. Hence $R = \Sigma F$. Its *position* is found from the necessary fact that the moment of R about any axis is equal to the algebraic sum of the moments of its components about the same axis; that is: $Rx_r = \Sigma(Fx)$.

$$\text{or } x_r = \frac{\Sigma Fx}{R} = \frac{M}{R}$$

This is really the same as was shown in **18**, when we combined a force and a couple. Every moment comes from a couple, and in all statical problems that couple is always easily found.

Ex. 1. Suppose a floating platform is to be acted upon by a system of parallel forces in a horizontal plane which do not balance, and that we are to find the *balancing* force.

In order to find the required balancing force, or balancing couple, find the resultant; and then balance it. Let the platform, Fig. 43, be rectangular, and let the given forces all act parallel to the edge OY, and let their distances from OY be known The data and calculation may be tabulated as shown in the table. Each force is resolved into a force along OY and a couple, as explained in **19**.

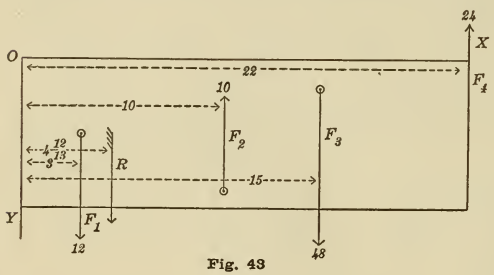


Fig. 43

The forces at O are then added, and the moments about a vertical axis OZ, are also added.

So that $R = \Sigma F$
and $M = \Sigma(Fx)$

The resultant force along OY is $+26$, and the resultant moment is $+128$. The resultant force and the resultant couple combined (see **18**) give a single force $+26$ at a distance from O equal to $M/R = 4\frac{12}{13}$. Hence the single resultant of the given forces is a force of 26 lbs. acting in a positive direction and at a distance $x_r = 4\frac{12}{13}$. It follows that the *balancing* force is 26 lbs, and that it acts in the *negative direction* along the line of R. If this force, F_5, were added to the given forces, the table would show complete equilibrium, since we should have $R = 0$ and $M = 0$.

F	x	Fx
$+12$	3	$+36$
-10	10	-100
$+48$	15	$+720$
-24	22	-528
$R = 26$		$M = 128$

Ex. 2. Find the resultant of the following system of forces applied to a similar body with opportunity for negative values of x. It will be found that the *resultant* is a couple, and that only a couple can balance the given forces. See Table.

F	X	M
$+40$	$+4$	
-16	$+6$	
-36	-2	
$+12$	-5	

51. Parallel forces in space. It is easy to conceive a rigid body acted upon by a variety of other bodies along parallel lines, with concentrated forces and lines of action so adjusted and proportioned that the body is at rest. In fact, every car that stands on the track, every house that stands on posts or columns, every table that stands on legs, every loaded wharf that stands on piles, furnishes an illustration of this proposition.

Since the forces balance, there must be no resultant tendency to move the body in any direction. This gives us one equation between the magnitudes of the forces:

$$\Sigma F = F_1 + F_2 + \ . \ . \ . \ F_n = 0$$

In like manner, since there can be no resultant tendency to turn the body about any axis in space, we have another general equation involving both magnitudes and directions: $\Sigma M = 0$, the axis of M being any line in space; more especially a line perpendicular to the given forces.

It was shown in **35** that a moment about any axis in space could be resolved into three component moments about three rectangular axes; that is

$$M = \sqrt{(M_1^2 + M_2^2 + M_3^2)}$$

if the axes referred to are OX, OY and OZ. These axes may be taken

at will. Let OZ be parallel to the common direction of the lines of action of all the forces. It follows that

$$M_3 = 0$$

inasmuch as a force can have no direct tendency to turn a body about a line parallel to its own line of action. There remains therefore $M = \sqrt{(M_1^2 + M_2^2.)}$ Now, M cannot be zero, unless both M_1 and M_2 are zero. Hence the "Equations of Equilibrium" are

$$\left.\begin{aligned} \Sigma(F) &= 0 \\ M_1 &= 0 \\ M_2 &= 0 \end{aligned}\right\}$$

52. The choice of co-ordinate axes. In accord with what appears to be the best usage, in all space problems in this book, the axis OZ will generally be taken *vertical*; the axes OX and OY are so taken that positive rotation about OX looking from $+X$ towards O carries $+Y$ into $+Z$; and a positive rotation around OY carries $+Z$ at once into $+X$. It is well to note the sequence of letters and numbers: X, Y, Z; 1, 2, 3. Rectangular axes will accordingly stand thus: (Fig. 44). All the co-ordinates of P are shown positive. Arrows on the axes show how one *must look* at a couple, or moment.

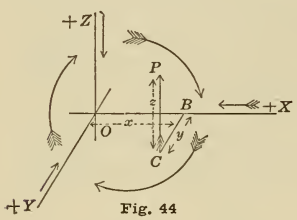


Fig. 44

An examination of Fig. 44 will disclose the fact (shown again for emphasis) that Fy is a right handed moment to an observer looking from $+X$ towards O, whenever F and y are both positive, or both negative; and that the moment is left-handed or negative whenever the factors have different signs. Accordingly, the algebraic law of signs holds for every term in $\Sigma(Fy)$.

It is just the reverse for every term in the expression $\Sigma(Fx)$. When the algebraic product is positive, the moment is left-handed and therefore negative; and when the algebraic product is negative the moment is positive.

Hence
$$\left.\begin{aligned} M_1 &= \Sigma(Fy) \\ M_2 &= -\Sigma(Fx) \end{aligned}\right\}$$

This apparent anomaly of a negative product from positive factors is unavoidable.

Ex. Take as an illustration an ideal system of balanced forces which are parallel and may be tabulated as follows:

The forces represented in the table are fairly shown in this space drawing, Fig. 45. The points of application are assumed to be in plane XY. The student may combine these forces by pairs according to any method already given, and then combine the partial resultant until he finds remaining two resultant forces *which will directly balance* each other. This checking process should not be neglected. The forces shown should be laid off with great accuracy.

F	x	y	Fx	Fy
+8	+2	+3	+16	+24
−6	+3	−1	−18	+ 6
+4	−1	0	− 4	0
−3	+2	+5	− 6	−15
+2	+1	−5	+ 2	−10
−5	−2	+1	+10	− 5
$\Sigma F = 0$			$\Sigma(Fx) = 0$	$\Sigma(Fy) = 0$

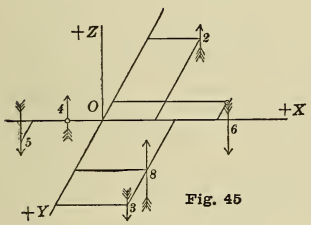


Fig. 45

53. The equations of equilibrium. It is readily understood that one may know that a system of parallel forces balances, yet be ignorant of the numerical values of some of the quantities which enter into the equations of equilibrium. The three equations of (**51**) will suffice to find three unknown quantities, provided every equation has at least one unknown quantity in it.

There are three general cases each of which will be illustrated. The three unknowns may be:—

1. Three unknown forces.
2. Two unknown forces and one co-ordinate.
3. One unknown force and two co-ordinates—one an x, and the other a y.

Case I. **Three forces are unknown.** Suppose a rigid slab, rectangular in shape, is supported in a *horizontal position* by four vertical wires connected with adjustable pins, and that one of the wires includes a spring balance which shows the tension; the other three tensions are to be found. All the points of attachment with the slab are known.

Let the length of the slab be twelve (12) feet, and the width four (4) feet, and assume that the slab weighs

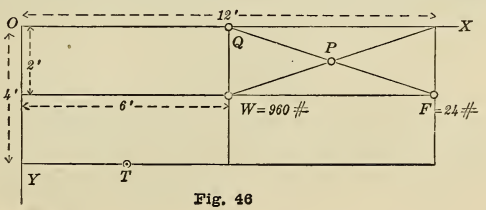


Fig. 46

960 lbs. The axes assumed, and the position of the four supporting forces are given in Fig. 46. The known force is F. The three equations of equilibrium are

$$P+Q+T+24-960=0 \tag{1}$$

$$M_1=24\times 2+P\times 1+T\times 4-960\times 2=0 \tag{2}$$

$$M_2=960\times 6-24\times 12-P\times 9-Q\times 6-T\times 3=0 \tag{3}$$

Eliminate Q from (1) and (3) and get (4).

Eliminate P from (2) and (4) and get the value of T, and so on.

Ex. 1. Having fully mastered and completed the above, let the student take new axes with O at the center of the slab.

Ex. 2. Let him invent a new example with two weights and five forces, three of which are unknown.

54. Case II. **Two forces and one co-ordinate are unknown.**

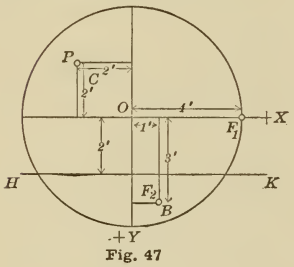


Fig. 47

Problem 1. Suppose a circular plate weighing 420 lbs., 8 feet in diameter, of uniform thickness and density, to be supported in a horizontal position by four wires connected overhead as in the last illustration. See Fig. 47. Two of the forces are known, F_1 and F_2; but two, P and Q, are unknown. $F_1=80$ lb and acts at A. $F_2=120$ lb. and acts at B. One unknown force, P, acts at C ($x=-2$, $y=-2$). The unknown force Q acts somewhere on the line HK, whose equation is $y=2$. Required the magnitudes of P and Q, and the x-co-ordinate of Q.

Ans. $P=200$ lb, $Q=20$ lb, $x=-2$.

2. A table is 8 feet square. It has three legs under the points A, B and C which are on the perimeter of a 6 ft. square as shown in Fig. 48. Two weights are on the table: one, W_1 is at the center; the other is somewhere on the line AD. The leg A supports 80 lb; B supports 30 lb.; C supports 48 lb. Find W_1 W_2 and the position of W_2.

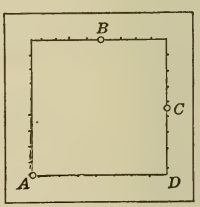


Fig. 48

55. Case III. **One force and two co-ordinates an x and a y are unknown.** (It is immaterial which x and which y are unknown). The data may be tabulated as shown on the next page.

It is *assumed* that P, x and y are all positive, tho this assumption may prove false. In addition to what is given in the Table we *know* that they *balance*; hence the three summations are separately zero.

Find P, x and y, and make a space drawing showing the five forces.

It is thus seen that the Equations of Equilibrium enable one to find the values of three unknowns. A fourth cannot be found unless there is some additional condition which may lead to a fourth equation, such as: $T=hP+kQ$, or, $x=ay+c$.

It will be seen later, when treating of *elastic* bodies, that some times a fourth independent equation can be found.

F	x	y	Fx	Fy
60	3	6		
−40	−2	8		
−104	4	$+y$		
50	−1	0		
P	$+x$	−4		
0			0	0

56. To find the resultant of a system of known parallel forces in space, which do not balance.

It is evident at once that the Equations of Equilibrium do not apply. It is also evident that the resultant force is the algebraic sum of the given forces. It is equally evident that if the positions of the forces are given in rectangular co-ordinates, the resultant will at *first* appear as a force acting thru the origin, along the axis OZ, and *two moments*: one about OX and the other about OY.

That is, we shall have:—

$$R=\Sigma F=F_1+F_2+F_3+\text{etc.}$$
$$M_1=\Sigma(Fy)=F_1y_1+F_2y_2+\text{etc.}$$
$$-M_2=+\Sigma(Fx)=+F_1x_1+F_2x_2+\text{etc.}$$

If x_1 and y_1 are the co-ordinates of a point in the line of action of R in space, we shall have

$$-M_2=Rx_1=\Sigma(Fx);\ \text{so that}\ x_1=\frac{\Sigma Fx}{R}$$

$$M_1=Ry_1=\Sigma(Fy);\ \text{so that}\ y_1=\frac{\Sigma Fy}{R}.$$

F	x	y	Fx	Fy
8	6	4		
−10	2	−3		
12	3	+7		
6	−4	+8		
−14	−5	−6		
Σ			Σ	Σ

This is the general case. It may be illustrated by an ideal example. The data are most conveniently given in tabulated form.

Show that the resultant is a force $+2$, parallel to OZ, whose line of action intersects the plane XY in the point:

$$x=55$$
$$y=139$$

It goes almost without saying, that a force of -2 parallel to OZ, thru the same point would *balance* the above system.

57. Remarks as to the resultant. 1. If $\Sigma(Fy)=0$, there is no moment about the axis OX; this shows that the resultant force intersects that axis.

2. If both $\Sigma(Fy)$ and $\Sigma(Fx)$ are zero, the resultant force acts thru O, and along the axis of Z.

3. If $\Sigma(F)=0$, the resultant is not a single force, but is a moment (or a couple of forces).* It will be well to illustrate the last remark.

58. When the resultant of a system of parallel forces in space is a couple.

1. Ex. The Table represents a set of data, and the calculation.

F	x	y	Fx	Fy
+8	+ 6	+5	+48	+40
− 6	− 2	+3	+12	−18
+3	0	−8	0	−24
− 5	+10	−7	−50	+35
$R=O$			+10	+33

Hence $M_2=-\Sigma(Fx)=-10$ See (**52.**)

$M_1=+\Sigma(Fy)=+33$

Since M_2 is negative it is laid off on the *negative* part of the Y-axis, pointing towards O, see **25**. In this case θ is a negative angle, or a positive obtuse angle. (Fig. 49).

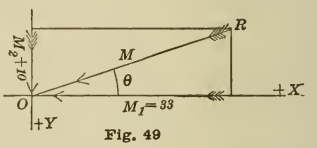


Fig. 49

The *resultant couple* acts in a plane perpendicular to the line RO, with a moment $\surd(1089)$, which is right-handed to an observer looking from R towards O.

2. A second illustration, in which both M_1 and M_2 are positive.

F	x	y	Fx	Fy
15	+3	+2	+45	+30
− 6	+8	+10	−48	−60
+ 4	−4	+12	−16	+48
−13	−1	0	+13	0
0			− 6	+18

$M_1=\quad\Sigma(Fy)=+18$

$M_2=-\Sigma(Fx)=+\ 6$

$M\ =\surd(36+324)$

$\tan\theta=1/3$. See Fig. 50.

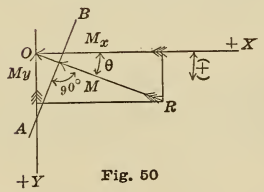


Fig. 50

The resultant couple is in the plane AB.

Exercises.

Let the student invent and solve two problems in one of which M_1 and M_2 have opposite signs; and one in which both M_1 and M_2 are negative.

* The formulas give, when $R=O$, $x_1=\infty$ and $y_1=\infty$, yet $R\times x_1=o\times\infty=\Sigma(Fx)$ which is a finite quantity. Also $R\times y_1=o\times\infty=\Sigma(Fy)$ which is *another* finite quantity. The student will see here an illustration of a truth which may have seemed obscure in Algebra.

CHAPTER IV.

Co-Planar Forces with Center Lines of Action Meeting at a Point.

59. Case I. **Three forces balance.** Since two non-parallel forces acting on a body cannot possibly balance, there must be at least three, and any one must directly balance the resultant of the other two. This fact gives rise to very important relations between the balancing forces which cannot be clearly stated till we have found how two intersecting forces may be combined, or replaced by a single force.

1. It is important that the reader picture in his mind the physical conditions of the general problem. *One* body is to be acted upon by *three* other bodies; each action is the resultant of a more or less distributed force; these force lines representing concentrated actions must lie in a plane; and the lines of action meet at a point.

2. For example, suppose a heavy solid body, B, *rests* against the combined action of the earth, the *smooth* end of a block P, and the tension in a wire rope Q. The force W is the resultant of an attractive force distributed thru a *volume*. The force P is the resultant of a pressure distributed over a *surface*. The force Q is a tension sent along a wire from an overhead hook *in the plane of* W and P; the three lines of action meet at O. The situation being *ideal*, all other actions (atmospheric and magnetic) are omitted. P is acting *towards* O; W and Q act *from* O. However, the forces are equally well represented in Fig. 52, as acting *from* their common point.

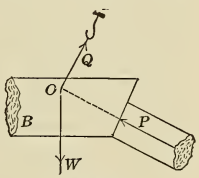


Fig. 51

60. **The parallelogram of forces.** If two intersecting forces are represented by two right lines drawn to scale *from* or *to* their common point, and if on them as sides, a parallelogram be drawn, the diagonal drawn *from* or *to* the common point represents their resultant. A great many proofs of this proposition have been given, some of which may be found in cyclopedias; only one is here given.

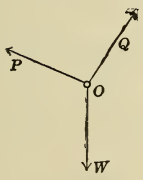


Fig. 52

Let the given forces be P and Q, both acting from O. Fig. 53. It is to be shown that the force R represented by the diagonal of the parallelogram constructed on P and Q is their resultant. Let C be

any point in the plane, and let a perpendicular to the plane at C be an "axis of moments." It will now be shown that the moment of R about the axis at C is equal to the sum of the moments of P and Q about the same axis. Draw OC, and connect the arrow heads A, B and D with C; finally draw AE and BF parallel to OC. The force lines being drawn to scale, the other lines must be considered as drawn to the scale.

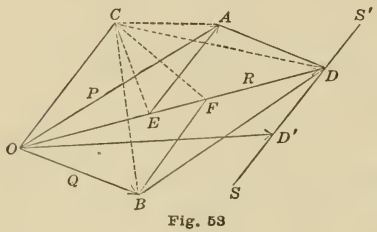


Fig. 53

The moment of R with respect to C is $R\times$ an arm from C to OD. or: moment of $R = 2\times$ area of $\Delta\ OCD$.

Similarly, the moment of $P = 2\times$area of $\Delta\ OCA$.
$= 2\times$area of $\Delta\ OCE$.

and the moment of $Q = 2\times$area of $\Delta\ OCB$.
$= 2\times$area of $\Delta\ OCF$.

Now, from the equal triangles OAE and BDF it is seen that $OE = FD$, and hence the area of the triangle $OCE =$ the area of the triangle FCD.

Hence the moment of $P = 2\times$area of $\Delta\ FCD$.

And since the area $OCD =$ area of $OCF +$ area of FCD, it follows that *the moment of R is equal to the sum of the moments of P and Q.*

But the proof that R is entitled to be called *the* Resultant is not complete, since there are any number of forces acting thru O which have the same moment. For example, draw thru D a line SS parallel to OC. Let D' be any point on SS.

A force OD' will have the same moment about O that R has, since the area of the triangle $OCD' =$ the area of the triangle OCD. Moreover the moment of OD' about any other axis taken on the line OC or OC produced, will be equal to the sum of the moments of P and Q about the same axis. If, however, we choose an axis outside of the line OC, we shall have another line $S'S'$ thru D, and all the forces OD' will be rejected except the single force $R = OD$.

Hence, since the moment of R is always equal to the sum of the moments of P and Q for the axis C, and is the *only force* which thus represents them wheresoever the axis is taken, it alone is entitled to the name "Resultant."

61. The following graphical method of finding the resultant of two converging forces, which do not meet within the limits of one's drawing, is merely a generalization of the special case with parallel forces given in (**40**). Let P and Q be the co-planar forces which do

not meet on the paper. To find the line of action and magnitude of their resultant. Fig. 54.

1. Parallel to the head-and-tail line Ab (which need not be actually drawn) draw the head-line BD, and the tail-line aO.

2. Parallel to the other head-and-tail line, draw a head-line AD and a tail-line bO.

3. The intersection of the head-lines is the head, and the intersection of the tail-lines is the tail, of the resultant required.

The analytical proof of this is left to the student.

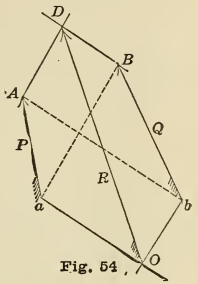


Fig. 54

62. The resolution of a given force. The given forces P and Q are known as the Components of R.

1. The converse of the proposition in **61** is, that any force may be resolved into two components by constructing a parallelogram on the line representing the given force as a diagonal. Thus Fig. 55, let R be a given force; its components may be R_1 and R_2, but they may as well be R_1' and R_2'; in short, there may be any number of pairs of components. By drawing a variety of parallelograms, the student should note the important fact that the resultant of two forces *may* be less than either of them; and that one or both the components of a force *may* be greater than the force itself. The relation depends upon the size of the angle $\phi = AOB$.

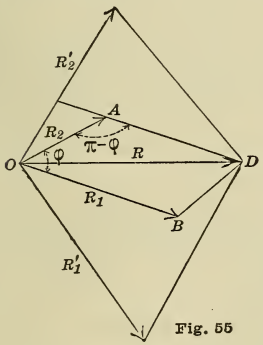


Fig. 55

2. The sides of the triangle OAD, may represent a force and its two components in magnitude and direction, but not fully in position; in fact, the position of all the forces might be changed as in Fig. 56, which means that the force R is equivalent* to the two components R_1 and R_2; and conversely, that the forces R_1 and R_2 are together equivalent to the single force R. As now understood, the lines of this triangle are frequently called VECTORS, α, β, γ and we read the equivalent equation

$$\alpha + \beta = \gamma$$

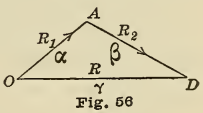


Fig. 56

3. Henceforth when we want the resultant of P and Q meeting at O, we shall not draw the full parallelogram, but we shall draw $OA = P$ and $AD = Q$; then OD will equal R.

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* *Equivalent* so far as the *motion* or *rest* of the body acted upon is concerned. It is not equivalent when internal stresses are concerned.

63. The resultant by trigonometry. When P and Q and ϕ, the angle between P and Q are given, it is evident from Fig. 57 that

$$R^2 = P^2 + Q^2 + 2PQ\cos\phi.$$

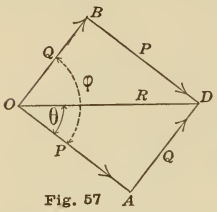


Fig. 57

When $\phi = \pi/2$ it is at once evident that

$$R = P\cos\theta + Q\sin\theta$$

and that $P\sin\theta = Q\cos\theta$, as well as that $R^2 = P^2 + Q^2$.

It is now seen that if the direction and magnitude of one component of a given force are given, the direction and magnitude of the other component are determined—graphically and by trigonometry.

64. The static triangle or the triangle of equilibrium. We are now prepared to bring in the third force which balances two given forces. It is evident that in order to balance the two given forces, it must balance their resultant. To do this it must have the same magnitude and be directly opposite. In other words, referring to Fig. 56, and remembering that the third force, which we will call S, is the exact opposite of R, and that if R was the vector γ, S must be the vector $-\gamma$; so that

$$\alpha + \beta = -\gamma$$
$$\text{or } \alpha + \beta + \gamma = 0$$

which is the vector equation of equilibrium. The triangle (of magnitudes and directions only) becomes (Fig. 58) the ***STATIC TRIANGLE*** or the *triangle of equilibrium.*

Whenever three forces balance, such a triangle can be drawn, and the relations of sides and angles can be found graphically or by trigonometry. The sufficient evidence of equilibrium is a state of rest, or of uniform motion. Cases of absolutely uniform motion are so rare that they are for the present neglected in our thought. The great value of the static triangle will be illustrated by a few examples.

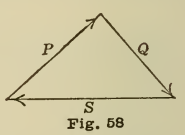


Fig. 58

65. Imponderable pins. 1. In dealing with centralized forces it is often convenient to assume that they act upon an imponderable pin, for thus we fix the attention upon a body (the pin) which is at rest under their action. This is particularly true when the forces arise from the action of wires, ropes, tie-bars, piers, posts, and struts, as at the joints of frames. In the case of an over-hanging hoist, such as is shown in Fig. 59, we may conceive of a pin of inconsiderable weight, acted upon by a tie-bar T (which may be double, consisting

of two thin bars instead of one thick one), a strut, S, or inclined post, and a chain or rope which may be connected with a pulley-block which has several plies.

2. *The pin stands still*; hence the three forces acting upon it balance, and we can draw the static triangle, since all directions are given in the figure or are known on the construction itself. If the weight, W, is known, we can draw the static triangle to scale. The values of S and T can be measured from the drawing or calculated. See Fig. 60.

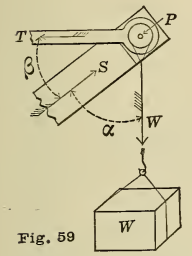


Fig. 59

If P represents the pin in Fig. 59, it is well to put the same letter within the static triangle, Fig. 60, and then follow and note the directions of the arrows which surround it. In every case the arrow shows the *direction* of the action in Fig. 59. For example: S in Fig. 60 points *upwards*; and it is therefore a strut in Fig. 59; T points towards the *left*; it therefore *acts* on P towards the left, and hence must represent a tie, or tension bar.

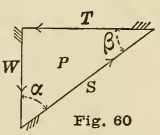


Fig. 60

3. If three non-parallel co-planar forces act upon a body at different points and *yet balance*, the three lines of action *must meet at* a point, or one could not directly balance the resultant of the other two.

66. Ex. 1. Take the ideal case of a heavy bar suspended against a vertical *smooth* wall. A weightless wire fastened to a hook or staple in the face of the wall at C supports the end B, Fig. 61, while the other end rests against the smooth wall which can act only in the direction of a normal to its surface. The bar and wire must be in a vertical plane normal to the wall. If G is the center of gravity action of the bar, the line of the wire must pass thru the intersection of the action lines of H and W at O. The *static triangle* can now be drawn showing the magnitudes of forces and angles. Fig. 61a.

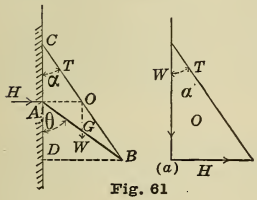


Fig. 61

2. If G be at the center of the line AB (as in the case of a uniform bar) it is seen that O is the middle point of CB, and $AD = AC$. Hence, when θ and l, the length of the uniform bar, are known, the position of C is found by laying off AC equal to $l\cos\theta$.

3. The reader must not fail to see that the above suspended body is in *unstable* equilibrium; the touch of a fourth force would cause it to fall or to collapse against the wall, even if it were prevented from swinging sidewise. If, however, the wall were rough (and all walls *are* rough), there would be a certain stability.

4. It is evident that the pull of the wire CB could be replaced by the thrust of a strut or prop SB, without in any way changing the static triangle; we would then let T stand for thrust instead of tension. It is also evident that a heavy weight may be hung at G without dislocating the bar, as it would only increase W.

67. A third example will further illustrate the principle that three balancing forces meet at a point. A slender uniform rod rests upon the inner surface and edge of a smooth hemispherical bowl. Fig. 62. We are to find the relation between l and θ.

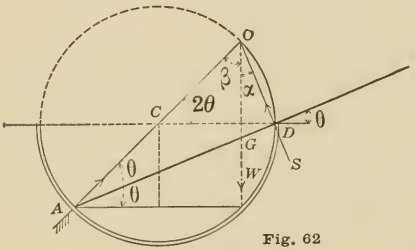


Fig. 62

The action at A must be normal to the spherical surface, and hence along a diameter; the action at D must be normal to the rod, as tho the edge were an infinitesimal round. Hence the gravity action must be in a line vertically below O, where the two normals intersect, and the length of the bar, which will rest at A and D, is determined, for its center must be at G.

1. If θ be the inclination of the rod to the horizontal, it is evident from the figure that $2r\cos 2\theta = \frac{1}{2} l\cos\theta$, which gives the necessary relation between l and θ. It is further evident that any movement of the rod due to a new force would have the effect of raising the point G, or lifting the bar, so that when relieved from the disturbing force it would slide back to its position of equilibrium. This is a case of "stable equilibrium."

2. The inferior limit to the length of the rod we have been discussing is when $l = 2r\cos\theta$. The *minimum length* of the rod is found by eliminating θ from the two equations

$$\left.\begin{aligned} 2r\cos 2\theta &= \tfrac{1}{2} l\cos\theta \\ l &= 2r\cos\theta \end{aligned}\right\}$$

Whence $l = 2r\sqrt{(2/3)}$ and $\cos\theta = \frac{1}{3}r\sqrt{6}$.

3. If l be less than $\frac{2}{3}r\sqrt{6}$. the bar will rest only when G is directly below C.

Problems.

68. 1. Given a smooth vertical wall, and at a distance s a lower parallel wall with a horizontal edge which is rounded and smooth.

Fig. 63. What is the inclination of a smooth, slender, uniform rod which will rest over the corner and against the wall? (Note where the balancing forces meet).

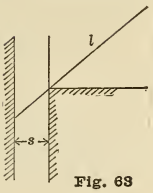


Fig. 63

Ans. $\cos\theta = \sqrt[3]{\frac{2s}{l}}$.

2. Find the stress in the members of this derrick due to the weight (8 tons). Fig. 64.

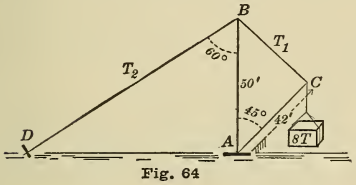


Fig. 64

Draw a static triangle for the pin at C, and then another for the pin at B.

3. If AB and AB' are equal smooth rods looped on a pin or ring at A, Fig. 65, prove that the condition, that the pair will *rest* on the surface of a smooth horizontal cylinder whose radius is r, is

$$l = \frac{r}{\sin^2\theta \tan\theta}$$

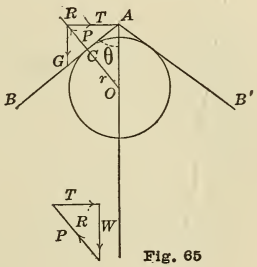


Fig. 65

4. A smooth bar rests upon two smooth inclined planes whose intersection at O is horizontal. Find the inclination of the bar to the horizontal. Fig. 66.

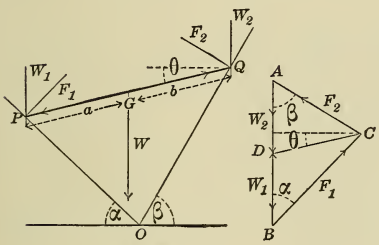


Fig. 66 (b)

The weight is centered at G. First draw the static triangle of external forces W, F_1 and F_2; Fig. 66, b; then resolve the weight into vertical components, W_1 at P, and W_2 at Q, and complete the static triangles for the points P and Q, thereby finding CD, the "*thrust*" in the bar, and its inclination.

From the diagram (b) prove that

$$\tan\theta = \frac{a\cot\alpha - b\cot\beta}{l}$$

5. A heavy cylinder is held against a smooth wall by a smooth rigid bar which is hinged upon a pin at A (Fig. 67), and held by a tie T to a hook at H, as shown. Given the weight and radius of cylinder, the length of bar, and all angles, to find the tension T, the pressures P_1 and P_2, and the magnitude and direction of the hinge action, due solely to the weight of the cylinder.

Solution. Three forces act upon the cylinder, hence the static triangle KEF whereby P_1 and P_2 are known. Fig. 67.

Since we omit the weight of the bar AB, only three forces are acting upon *it*, viz.: P_1, T and the hinge. We already have P_1, and we know the direction of T, and as the three must meet, say at D, we know the direction of the hinge action, P_3, namely in the direction AD. Its static triangle is therefore FEH. Thus the magnitude of T and both the magnitude and direction of the hinge action due to the cylinder are found.

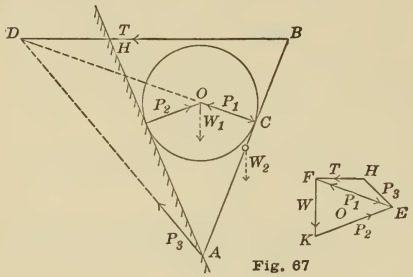


Fig. 67

69. How to determine the tension or compressive force in a bar of a loaded frame from the static triangles of its pins.

Let the members of a cantilever frame lie in a vertical plane with a load at the end. Fig. 68. We are to find the actions of the bars BA and CA upon the pin at A; and the actions of the bars AB, CB and DB upon the pin at B.

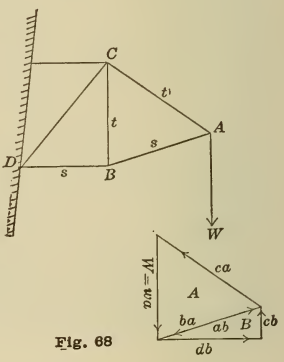


Fig. 68

Solution. The static triangle for A. The case is similar to that in **59.** Knowing one side of the triangle and the directions of the other two, it is quickly drawn. W acts *down*, BA acts *up*, parallel to the bar, and CA acts *up* to the left parallel to the bar CA, closing the triangle. The static triangle for A shows magnitudes and *character*. AB acts up on the pin A, and is therefore a *strut*; the bar CA also acts *up* and *away from* the pin, and therefore is a tie.

Going now to the pin B, three bars act on it and *balance*; hence B has a static triangle. The action of AB is known, being a strut it acts *down* on B, and the triangle already drawn gives it magnitude. We therefore use the line ba, or ab, and draw db for the bar DB, and cb for the bar CB, and mark the arrows of the static triangle for B. We note that the arrow db points to the *right*, showing that DB acts against the pin B, while the arrow cb points *up*, showing that CB acts *away from* the pin B. DB is therefore a strut, and CB is a tie, and they are lettered accordingly.

This is all very simple, but it cannot be too clearly seen. Occasions will surely arise when students will find it necessary to refer back to this section and go over it again.

70. Case II. **The polygon of forces and the static polygon.** When more than three co-planar forces are given with a common point in their lines of action, their resultant, or their necessary relations, if they form a balanced system, are readily found by an extension of what has already been given. Fig. 69.

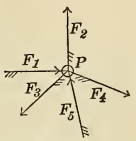


Fig. 69

Given five forces acting upon a pin at P. To find their resultant.

Graphical Solution.

Take any convenient point O as an origin; draw OA equal and parallel to F_1; draw AB equal and parallel to F_2; this gives OB equal and parallel to the resultant of F_1 and F_2.

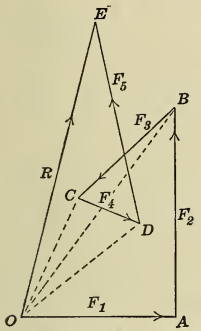


Fig. 69 (a)

Draw BC equal and parallel to F_3; OC is then equal and parallel to the resultant of F_1, F_2 and F_3.

Draw CD equal and parallel to F_4; OD is then equal and parallel to the resultant of F_1. . . . F_4.

Draw DE equal and parallel to F_5; OE is then equal and parallel to the resultant of all the forces, F_1 . . . . F_5.

The *order* in which the forces are taken is not important, the resultant is still the same. This statement should be put to the test by a careful drawing. It is evident that the lines OB, OC, etc., are of no use excepting always *the last one* which is the *resultant* sought. The line representing their resultant is always drawn *from* O to the *end* of the last component force.

Had the line of the last component ended in O, there would have been no resultant, as we should have had

$$R = 0$$

which is the evidence of a balanced system of forces. In all practical statical problems R is zero; and every polygon drawn as above, *which closes*, is called a Polygon or Equilibrium, or more conveniently a "Static Polygon."

71. Static triangles and polygons are useful in finding unknown magnitudes and directions. In every case where a resultant is to be found, or a static triangle or polygon is to be drawn, the data must be geometrically sufficient to that end. In most of the problems thus far solved, the magnitudes of two forces were found, all angles being given or determined by the geometrical necessity of convergence of the lines of action. In the last general problem several forces were fully given, and a single angle and force were to be found. The following problem is in a new class.

Ex. Given one force in position, a second force in magnitude only, and a third in direction only.

This may be illustrated by a special device using weights (Fig. 70) F_1, F_2 and F_3.

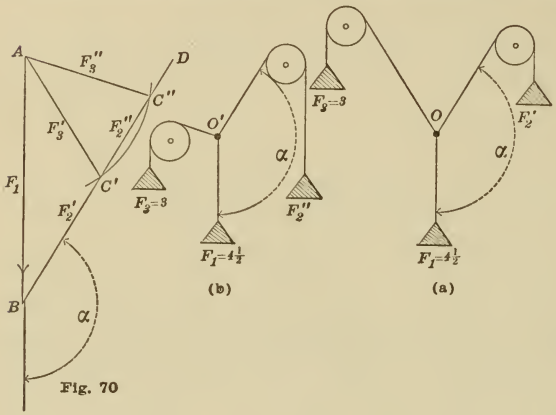


Fig. 70

Let O be the pin or body acted upon.

Let F_1 represent the force fully known $=4\frac{1}{2}$ acting down.

Let α be the known *direction* of F_2, relative to F_1; $\alpha = 150°$.

Let $F_3 = 3$ represent the magnitude of the third or balancing force.

We lay off to scale from any point as A, F_1 as AB to the point B; draw the direction of F_2 as OD; with F_3 as a radius, from A as a center, draw the *arc* $C'C''$. The intersections of this arc with the line BD determine the magnitudes of F_2' and F_2''. We get in general two solutions for the magnitude of F_2 and two directions for F_3. We see that F_1, F_2' and F_3' balance at O; 70*a*, and that F_1, F_2'' F_3'' balance at O'; (*b*).

72. Illustrations of the use of the static polygon.

Ex. 1. Four equal cylinders lie within a hollow semi-cylinder as shown in vertical end view. Fig. 71. Given the weights, to find the pressures P_1, P_2, P_3, P_4, P_5, P_6, and P_7.

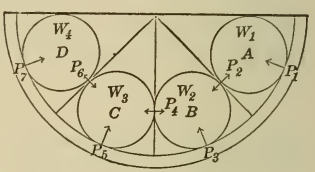


Fig. 71

Beginning with A, we draw the static triangle $W_1P_1P_2$. Fig. 72. We next draw the static polygon for B: $W_2P_2P_3P_4$. Then follows the polygon for C: $W_3P_4P_5P_6$. Finally, the static triangle for D: $W_4P_6P_7$, and the polygon closes. The reader will notice that since the four forces acting on B balance, they must, when drawn as vectors, form a *closed polygon*; hence P_4 returns to O. Having drawn Fig. 72 accurately, we should have no difficulty in checking the following:

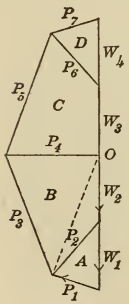


Fig. 72

$$P_7 = P_1 = W\sqrt{2 - \sqrt{2}}$$

$$P_6 = P_2 = W$$

$$P_5 = P_3 = W(1 + \sqrt{2})\sqrt{2 - \sqrt{2}}$$

$$P_4 = W\sqrt{2}$$

Ex. 2. A heavy horizontal cylinder, solid or hollow, rests upon two unequal cylinders which rest upon a horizontal bed with guard walls. Fig. 73. Weights being given, find the pressures at all points. The horizontal thrusts in AB and AC being balanced by retaining walls.

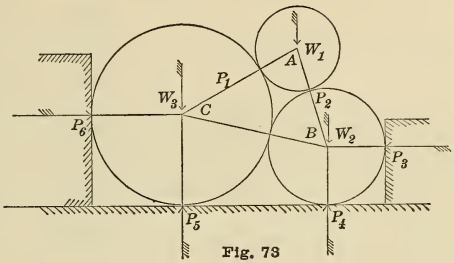


Fig. 73

The diagram (Fig. 74) is exceedingly simple, and values may be calculated when weights and radii are known. Of course it was evident without the drawing that

$$P_4+P_5=W_1+W_2+W_3,$$

and that $P_3=P_6$, as they were the external forces acting on the group.

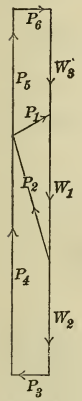


Fig. 74

73. The static polygon in "graphical statics." The Static Polygon enables us to find the magnitude of two forces when all other "parts" are known. This property is of the greatest value in finding the stresses in the members of frames when both frames and external forces may be regarded as co-planar. This deserves a somewhat extended illustration, as it will bring to us the very elegant method of "Reciprocal Polygons"* given in Chapter XI.

74. Equations of equilibrium and their uses. In every problem thus far solved by means of the static triangle or the static polygon, the data (forces and angles) were assumed as known, and the accuracy of the drawing was readily checked by trigonometry applied to the diagrams. When the known angles are the inclinations of forces to a given direction, that direction should be taken as the axis OX with OY perpendicular to it, and the rectangular components found by the method already given. Since the forces balance, their components along OX and along OY must balance independently, and their moment about *any axis* must be zero. Hence the Equations of Equilibrium

$$\Sigma F\cos\theta = 0$$
$$\Sigma F\sin\theta = 0$$
$$\Sigma Fl = 0$$

Here F is the representative of a force, known or unknown, and θ is

* So named by the eminent scholar Clerk—Maxwell.

its inclination to the axis X. For brevity we may write $F_1 = F\cos\theta$, and $F_2 = F\sin\theta$, so that the equations become

$$\Sigma F_1 = 0$$
$$\Sigma F_2 = 0$$
$$\Sigma Fl = 0$$

The angle θ is measured positively from $+X$, clockwise, to the head of the force arrow drawn *from O*.

The arc θ is always positive and its valve lies between 0° and 360°.

75. Equations for a resultant. When the given forces do not balance, the *resultant* is readily found by means of the following equations, which, after the above, are almost self-evident.

$$\Sigma F_1 = R_1$$
$$\Sigma F_2 = R_2$$
$$R = \sqrt{R_1^2 + R_2^2}$$
$$\tan\theta = R_2/R_1$$

All these quations are useful, especially those of **74.** The third equation $\Sigma Fl = 0$, is convenient when l is the perpendicular from a point on the line of one force to the line of action of another force.

Ex. 1. A rigid prismatic rod of small diameter stands on a rough horizontal plane leaning against a smooth vertical wall. Find the pressure against the wall. Fig. 75. Three forces are acting on the rod, *and they balance.* Hence their lines of action must meet at a point. Given the position and length of the rod, the point of convergence is seen to be O where H and W meet. If we take B as the axis for moments we see at once, since $\Sigma Fl = 0$,

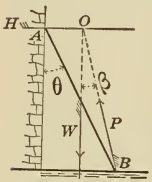


Fig. 75

$$Hl\cos\theta = \tfrac{1}{2}Wl\sin\theta$$
$$H = \tfrac{1}{2}W\tan\theta$$

But this value can be found from the equations $\Sigma F_1 = 0$ and $\Sigma F_2 = 0$.

$$H - P\sin\beta = 0$$
$$W - P\cos\beta = 0$$

Hence
$$H/W = \tan\beta$$

But from the figure $\tan\beta = \frac{1}{2}\tan\theta$, hence

$$H = \tfrac{1}{2}W\tan\theta, \text{ as before.}$$

This shows that *the three equations of* **74** *are not independent. They serve for the discovery of but two unknown quantities.*

Ex. 2. Find the resultant of the forces given in the table, both by graphics and by analytics; the forces all act at a point in OX.

F	θ	$\cos\theta$	$\sin\theta$	F_1	F_2
16	$\pi/4$				
20	140°				
10	210°				
25	$\frac{3}{2}\pi$				
		$R=$		$\theta_r=$	

76. Comparison of graphical and analytic methods. All of the problems we have had thus far in this chapter may be solved by equations of equilibrium. We shall have frequent use of these equations later on, and the student should finally be equally free to use the graphical method or what is called the analytical method. In earlier works on Mechanics, graphical methods were little used. For the sake of comparing the method by equations alone with the graphical method with trigonometry applied to the force polygon, we quote from good authorities solutions of two problems already solved.

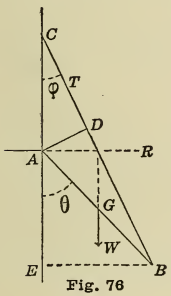


Fig. 76

Ex. 1. "A heavy uniform beam, Fig. 76, AB, rests with one end, A, against a smooth vertical wall, and the other end, B, is fastened by a string, BC, of given length, to a point, C, in the wall. The beam and the string are in a vertical plane; it is required to determine the pressure against the wall, the tension of the string, and the position of the beam and the string.

"Let $AG=GB=a$, $AC=x$, $BC=b$,

weight of beam $=W$, tension of string $=T$, pressure of wall $=R$.

then we have $BAE=\theta$, $BCA=\phi$

for horizontal forces, $$R-T\sin\phi=0 \tag{1}$$

for vertical forces, $$W-T\cos\phi=0 \tag{2}$$

for moments about A, $$Wa\sin\theta-(T.AD=Tx\sin\phi)=0 \tag{3}$$

$$\therefore\ a\sin\theta=x\tan\phi \tag{4}$$

and by the geometry of the figure

$$\frac{b}{2a}=\frac{\sin\theta}{\sin\phi} \tag{5}$$

$$\frac{x}{2a}=\frac{\sin(\theta-\phi)}{\sin\phi} \tag{6}$$

Solving (4), (5) and (6), we get

$$x = \left[\frac{b^2-4a^2}{3}\right]^{\frac{1}{2}}$$

$$\cos\phi = \frac{2}{b}\left[\frac{b^2-4a^2}{3}\right]^{\frac{1}{2}}$$

$$\sin\theta = \frac{1}{2a}\left[\frac{16a^2-b^2}{3}\right]^{\frac{1}{2}}$$

from which R and T become known." (Price's Anal. Mech's, Vol. I p. 69.)

Ex. 2. For the second example, showing the occasional simplicity of the analytic method, I quote a problem and solution from Professor Edward A. Bowser, Analytic Mechanics, 1884.

"A heavy beam, AB, rests on two given smooth planes which are inclined at angles, α and β, to the horizon; required the angle θ which the beam makes with the horizontal plane, and the pressures on the planes.

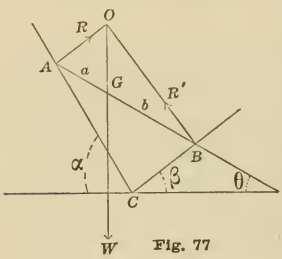


Fig. 77

"Let a and b be the segments, AG and BG, of the beam, made by its center of gravity, G; let R and R' be the pressures on the planes, AC and BC, the lines of action of which are perpendicular to the planes since they are smooth, and let W be the weight of the beam. Then we have

for horizontal forces, $$R\sin\alpha = R'\sin\beta \qquad (1)$$

for vertical forces, $$R\cos\alpha + R'\cos\beta = W \qquad (2)$$

for moments about G, $$Ra\cos(\alpha-\theta) = R'b\cos(\beta+\theta) \qquad (3)$$

Dividing (3) by (1), we have

$$a\cot\alpha + a\tan\theta = b\cot\beta - b\tan\theta$$

$$\tan\theta = \frac{a\cot\alpha - b\cot\beta}{a+b}$$

and from (1) and (2) we have

$$R = \frac{W\sin\beta}{\sin(\alpha+\beta)}; \qquad R' = \frac{W\sin\alpha}{\sin(\alpha+\beta)}.$$

77. The elegance and simplicity of the force diagram. Two perforated balls, A and B, whose weights are W_1 and W_2, slide down on two *straight smooth* rods until further motion is prevented by a weight-

less thread between them. The rods are stretched in a vertical plane from the same pin with equal inclinations α. Prove that

$$\tan\theta = \frac{W_1 - W_2}{W_1 + W_2} \cot\alpha.$$ (See Fig. 78),

and that the tension in the thread is

$$T = \frac{W_1 - W_2}{2\sin\theta}$$

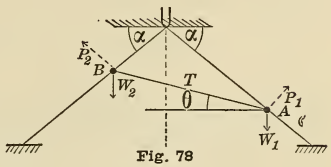


Fig. 78

Problems.

1. A triangle frame with pin-joints, whose sides are $5', 6', 7'$, stands vertically with its shortest side resting on a smooth level floor. What stress does a weight of 269 lbs. placed at the vertex cause in the lowest side? Draw the static diagram and measure the force line.

2. A camp stool (Fig. 79), consisting of four equal legs arranged in two oblique crosses, as AK and CH (with a horizontal connecting pin at O, and parallel bars connecting corresponding tops of the legs at A and C) carries a flexible band ABC from bar to bar as shown. The band supports a heavy rod of small diameter, B. Assuming that the stool stands on a *smooth* horizontal floor; that all the members of the stool are without weight, and that there is no friction at joints, we are to find the position of equilibrium, when the weight of the rod, and the lengths of legs and band are given.

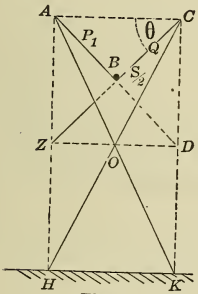


Fig. 79

78. The concentration of loads upon the pins of a frame.

RULE. WHEN A MEMBER OF A FRAME HAS WEIGHT, AND WHENEVER IT CARRIES CONCENTRATED LOADS, BOTH THE WEIGHT OF THE MEMBER AND THE LOADS ON IT MUST BE RESOLVED INTO PARALLEL COMPONENTS ACTING ON THE TWO SUPPORTING PINS.

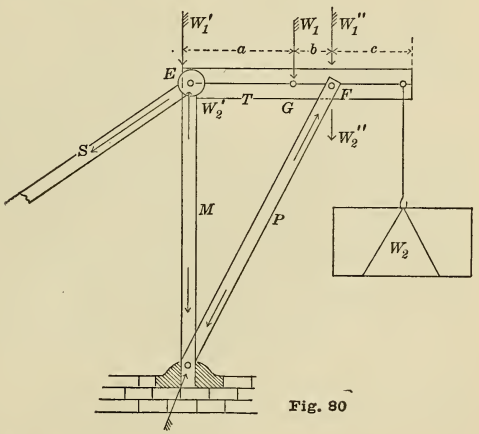


Fig. 80

The derrick shown in Fig. 80 has a cantilever or overhanging beam with a concentrated load, W_2, at the outer end. The weight of the beam, W_1, and W_2 are to be resolved into components at the two

pins, F and E. Then the static diagram can be drawn. G is the center of gravity of the beam; let W_1 be its weight, W_1' its component at E; and W_1'' the component at F.

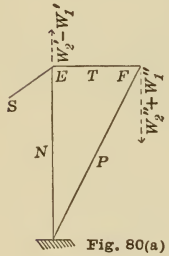
$$W_1' = \frac{b}{a+b} W_1, \qquad \text{and } W_1'' = \frac{a}{a+b} W_1$$

In like manner W_2 is resolved into

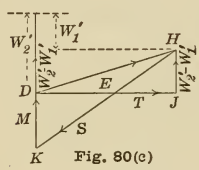
$$W_2' = -\frac{c}{a+b} W_2 \text{ at } E$$

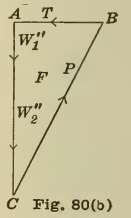
$$\text{and } W_2'' = +\frac{a+b+c}{a+b} W_2 \text{ at } F.$$

The *transformed derrick* now is as follows: The load at F is $W_2''+W_1''$ (Fig. 80*a*), and the static triangle for pin F is ABC (Fig. 18*b*), which gives the *thrust* P in the boom, and the *tension* in the beam T. The forces now known as acting upon the pin E are the horizontal tension T and the *negative* load $W_2'-W_1'$; the resultant of these two forces is shown in Fig. 80*c* to be the force DH acting upwards. The static triangle for pin E is now readily drawn, viz., DH *up* (already found); HK *down*, parallel to the *stay* S; and KD *up*, parallel to the mast N; giving for E the triangle of forces DHK; and all the forces or stresses are found.

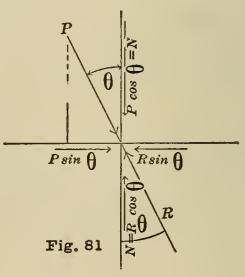


If instead of the resultant DH we had used its components T and $(W_2'-W_1')$, see Fig. 80*c*, we should have had the *Static polygon* of four sides, $DJHK$. We shall soon have static polygons in abundance. If all angles are given, the magnitude of the forces, in stay, mast and boom can be calculated by trigonometry.





79. Stability due to friction. 1. When a light bar, like a cane, a ladder, prop, or strut of any kind, transmits a thrust against a rough plane surface, the roughness of the plane tends to hold the end of the bar against slipping. Fig. 81. If the thrust is normal to the plane, there is no tendency to slip and the roughness is not needed to keep the bar in position. When, however, the thrust is inclined to the normal like P, Fig. 81, the resistance R must also be inclined, so that if it also be resolved into rectangular components, one of which is normal and one tangential, *along*



the plane, the latter is $P\sin\theta$, and the normal component is $P\cos\theta$. The ratio of the tangential to the normal component is

$$\frac{P\sin\theta}{P\cos\theta}=\tan\theta$$

2. If the inclination (to the normal) be gradually increased, a value of θ will soon be reached *beyond which* the roughness of the plane will not suffice to balance the tangential component of P, so that if θ is *made any larger*, the bar will slip and fall. The *static angle* is therefore limited, and the limiting value of θ is called ϕ. The force thus exerted by a rough plane in the direction of the plane itself is called *Friction*, and ϕ is called the "Angle of Repose" or the "Angle of *Friction*," or perhaps the "Static Angle." The roughness of surfaces both of bars and planes varies greatly.

3. Another way to illustrate friction is to place a rough body upon a rough inclined plane, Fig. 82. If it stands still, the forces balance. The normal component of the weight is balanced by the normal action of the plane; $W\cos\theta$. The tangential component of W is $W\sin\theta$, and this must be balanced by the friction.* If θ is small, very little friction is utilized or developed; as θ is increased more friction is needed. When $\theta=\phi$, the limiting value of friction is reached for that body, and the body is upon the point of sliding. If $\theta>\phi$ it *will* slide if a little initial sticking is overcome by a slight disturbing force. At the "static angle"

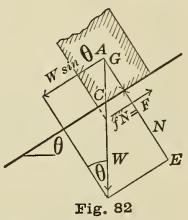


Fig. 82

$$Friction=W\sin\phi=\tan\phi\ (W\cos\phi)=N\tan\phi=fN$$

When $\theta>\phi$ the component of W down the plane becomes greater, while the total friction usually becomes less, since $N=W\cos\theta$ is less. Hence there is no longer a static body.

4. The factor $\tan\phi$, standing with or before N, is called f, the "Co-efficient of Friction," which is used when the body is actually moving. When a body is at rest and in no immediate danger of moving, the frictional action is *less* than $N\tan\phi$, and it may be zero. It is, however, of the highest importance that the *direction* of the oblique force which is to be balanced by friction be known and provided for. The following problem will illustrate this point.

---

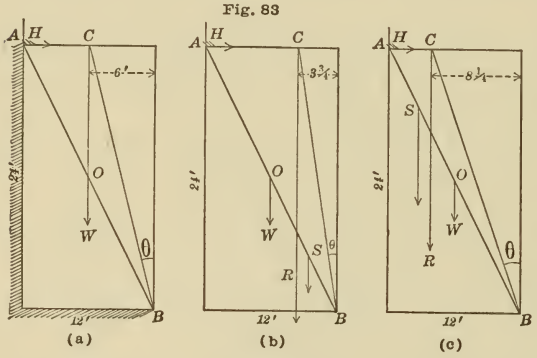
* The reader will observe that the component of W parallel to the plane acts thru G, and forms with the friction, fN, a couple which when combined with $W\cos\theta=N$ brings the *center of pressure* to the point C.

Problem.

80. A ladder, weighing 60 lbs, stands on a rough level floor, leaning against a smooth wall at a point 24 feet above the floor, with its foot 12 feet from the wall. Fig. 83. If $\phi = \arctan 1/3$ be the "angle of friction" between the ladder's feet and the floor, how far up the ladder can a man weighing 180 lbs. go before the ladder will slip and fall?

1. Before the man steps on the ladder, the line of action against the ground is not straight down the ladder; it is nearer the normal, as seen in Fig. *a*, where $\tan\theta = 6/24$. The three forces acting on the unloaded ladder are: a horizontal force AH; a vertical force $OC = 60$ meeting AH at C; the *third* force, CB, namely the action of the floor, *must pass* thru C; its obliquity is θ.

Fig. 83



2. When the man steps on the ladder, the point C is over the center of gravity of the two weights, the ladder and the man. Hence, it follows that, *at first* C has moved out, and the ladder is less likely to slip. So long as the man is on the *lower half* of the ladder, it is stable. Thus, when the man stands on a rung six feet from the floor, at S (Fig. *b*) the center of gravity of the two weights is 8.25 feet from the wall, and

$$\tan\theta = \frac{3.75}{24}$$

3. When the man is at the middle, $\tan\theta = 6/24$, as when the ladder was unloaded.

As the man mounts higher, the center of gravity moves to the left as does the point C, where the three forces always meet.

When the man reaches the point S (Fig. *c*), three-fourths of the way to the top (supposing that possible), the center of gravity of the two weights (and therefore the point C) has moved to a point only 3.75 feet from the wall, and $\tan\theta = \frac{8.25}{24}$, which is *more than* $\tan\phi$. Hence the ladder slips and falls just when the man is *about to reach* a height of 18 feet.

4. Experienced workmen know well that ladders are apt to slip if a man (and the heavier he is the greater the danger) goes too near the top, and by sad experience they have learned to secure the foot of the ladder, either by a block or tie, or by putting another man on the ladder near the foot so as to bring the center of gravity of the whole weight away from the wall thereby diminishing the angle θ.

5. Workmen know, too, that a light ladder is more likely to slip than a heavy one. If we were to make the unreasonable assumption that the ladder has no weight worth considering, the point C in the line of action of the supporting reaction would always be directly over the head of the man as he mounts the ladder, and in a horizontal line from the top of the ladder.

6. If the student will draw the static triangle for each of the three positions of the man on the ladder, he will see how H increases as the man mounts.

81. While friction is a great hindrance to the motion of bodies, it is a great aid to the stability of bodies at rest.

1. It was assumed above that the wall was absolutely smooth. Hence it could offer no resistance to the slipping down of the top of the ladder. In reality, as all walls are rough, a tendency to slide down would have developed an *upward force* equal to $H\tan\phi_2$, (ϕ_2 being the "Angle of Friction" for the ladder and wall), which would have had the effect of moving the center of gravity of the weights *away from* the wall thereby diminishing θ (and perhaps saving a fall).

The student should clearly see how much $H\tan\phi_2 = H/6$ would move C *away* from the wall provided the resultant load was acting at a distance of 3.75 feet from the wall. This matter is worth a bit of careful study. In fact, the value of H depends *somewhat* upon the friction between the ladder and the wall.

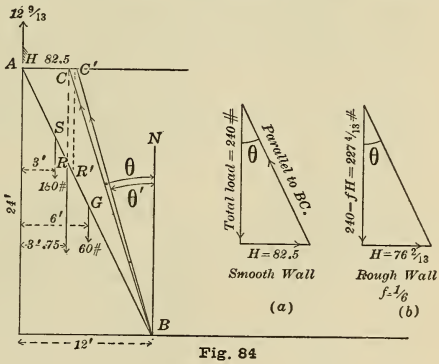


Fig. 84

2. The center of the ladder (Fig. 84) is at G with a man at S. The resultant load is at R, 3.75 feet from the wall. When the wall is smooth the point C is 3.75 feet from the wall and the reaction of the floor is along the line BC so that

$$\tan\theta = \frac{8.25}{24} = 0.344$$

Figure (84) (*a*) is the static triangle for the ladder, the value of H being

$$240 \times \frac{8.25}{24} = 82.5 \text{ lbs.}$$

If, now, we suppose the wall is rough, we shall have the co-efficient of friction $f = \tan \phi_2$. If we suppose the ladder is upon the point of slipping, the frictional action of the wall, which is *up*, is fH', so that we now have a new force involving H', acting upon the ladder. To find the value of H' under new conditions, take moments about B

$$24H' + 12fH' - 6 \times 60 - 9 \times 180 = 0$$

from which we get

$$H' = \frac{1980}{24 + 12f} = 76\tfrac{2}{13} \text{ lbs.}$$

if f is 1/6.

This gives the value of H' less than former H, so that the static triangle is new, the base being the new value H', and the vertical line being $240 - fH'$ the new θ' is seen to be less than the former value of θ.

$$\tan \theta' = \frac{76\frac{2}{13}}{227\frac{4}{13}} - 0.335$$

The movement of the point C to C', due to the friction on the wall, is readily seen if we get the resultant of the two vertical forces, namely 240 lbs at R, acting *down*, and H' acting *up* at the point A. By what was given in a previous chapter, the resultant is the force acting at the point *below* R, with a magnitude of 240 lbs. $- fH'$, giving the new point C' graphically. In fact, it is enough to prove the point stated above, namely, that the point C is carried *further away from the wall by the roughness of the wall*, since the resultant is sure to be beyond or below the point R. *We thus see that the stability of a body, like the leaning ladder, may be secured by means of friction at its base, assisted by friction at its top.*

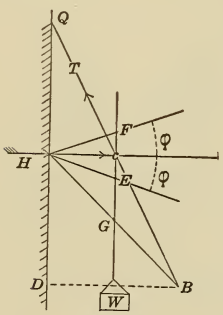


Fig. 85

82. Again, to use a modified form of the problem already solved twice ( ) of a uniform bar suspended against a vertical wall. It was found that when the wall was smooth, the cord or wire should be made fast to a hook as much above H as H was above D. Fig. 85. If now the wall be rough with an "angle of friction" ϕ, the bar (and its load hung at G) will be in equilibrium if C (where the three force lines meet) be *anywhere between E and F*; tho the *safest place* is at the center, C.

83. The concentration of distributed forces and the resolution of concentrated forces. When a heavy homogeneous and uniform timber has a support directly under or over its geometrical center, Fig. 86, we say it is in equilibrium, because its center of gravity is supported. We may say more explicitly that the resultant of the earth's attraction, which is distributed thru the whole mass of the beam, acts thru the center, and that the resultant is balanced by the support. But one may ask, how does the attraction on the vertical end layer get to the center? We answer that the first layer hangs by cohesion on the second layer; and that upon the third; and so on, each layer holding up all between it and the end, and being held up with its load by the one next to it on the inside. This vertically-acting force, made possible by cohesion, has been called "**shearing stress.**" Like friction, it is tangential, but unlike friction, it is far more intimate and continuous, the molecular interlocking being nearly perfect instead of imperfect and haphazard. By means of this **shearing stress,** the weights of all parts of the timber are brought to the center. The shearing stress is evidently the greatest at the center. The fact that this transfer of weight develops other stresses within the timber does not now concern us, tho it will concern us later on.

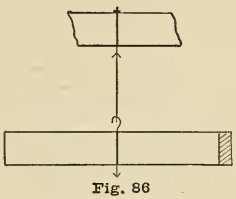


Fig. 86

84. If the timber has two supports, Fig. 87, the weight is concentrated at two points. The simplest method of making this distribution is by first concentrating the whole at the center of gravity, and then resolving W into two parallel components at A and B, in accord with the principles of Chapter III.

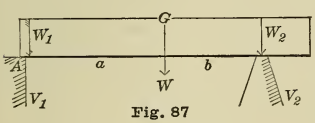


Fig. 87

Since the moments of the components are together, equal to the moment of their resultant about any axis, take an axis at A to find the component at V_2; then

$$Wa = W_2(a+b)$$

hence V_2, which must directly balance W_2, is found to be (in magnitude)

$$V_2 = \frac{a}{a+b} W$$

In like manner

$$V_1 = \frac{b}{a+b} W$$

85. Sometimes the method of support brings into a member of a frame a direct stress which is independent of the shear and bend-

ing stresses, as in this simple case. Suppose this heavy timber is supported by two separate cords or chains which are connected to a common hook. Consider the chains and timber as forming a triangular frame, with the weight of the timber (and whatever it may carry) concentrated at the ends into W_1 and W_2 with a single support at the hook. The static diagram can be drawn in the now familiar way, so arranged that only four straight lines exhibit the static triangles for the pins A and B. The cords are subject to direct tension, the timber is subject to a *direct thrust*, and to shear. Fig. 88(*a*).

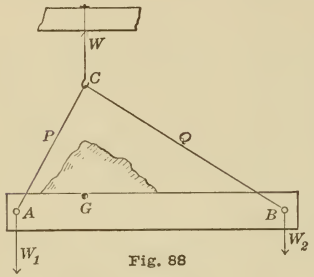


Fig. 88

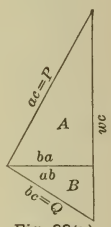


Fig. 88(a)

Had the timber and its load been supported by two chains to *separate* hooks, the timber *might have been* subject to direct tension.

The student may so represent a timber and draw its static (or stress) diagram.

CHAPTER V.

Non-Concurrent, Co-planar Forces.

86. In order to determine fully the force or couple required to balance a system of co-planar forces which act in different directions and at different points of a rigid body or frame, the resultant of the given forces is first found. There are four rather distinct methods of doing this. Every force must be actually or ideally given in magnitude, direction and position.

I. **Graphical solution.** Fig. 89. Combine the forces in pairs by producing their lines of action, if need be, till they intersect, and using the parallelogram, or by the methods of IV., 2 and (3). Combine the partial resultants in pairs, and so on till the single resultant is found. A little forethought in forming pairs will generally keep the work within bounds. The student should take his data from what is *given* in Fig. 89, as to the forces (1), (2), (3) and (4), their directions and their relative positions. The axes OX and OY are not needed in this method.

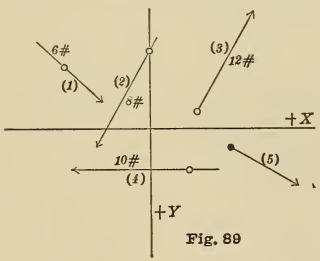


Fig. 89

If the final pair should consist of two equal and directly opposite forces, the resultant is zero, and the given forces balance.

If, in any such problem, the last pair should chance to form a couple, the combination can go no further; *the resultant is that couple*, and the system cannot be balanced by any single force; only an opposite couple can prevent the body from turning.

87. II. **Analytic solution.** Choose a pair of co-ordinate axes which can conveniently be used in locating points of action and directions of forces. Let the direction be determined by the angle θ, as shown in Fig. 90, and let a point of application be given by x and y.

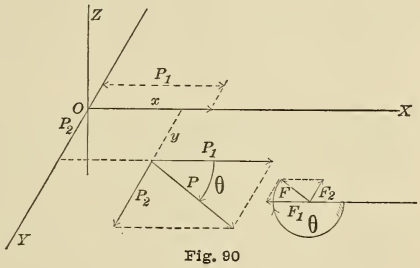


Fig. 90

(*a*) Reslove every force into its components $P\cos\theta = P_1$, and $P\sin\theta = P_2$. See Fig. 90.

(*b*) Resolve every *component* into an equal and parallel force at O, and a couple whose moment (about OZ) is $+xP_2$ for one component, and $-yP_1$ for the other. (Fig. 90.) Thus the *resultant* moment for a force P is $xP_2 - yP_1$. In the same way every given force, in general, is resolved into two forces, P_1 and P_2, at O, and a moment about OZ.

(*c*) Summing results

$$R_1 = \Sigma P_1 = \Sigma P\cos\theta.$$
$$R_2 = \Sigma P_2 = \Sigma P\sin\theta.$$
$$M = \Sigma(xP_2 - yP_1) = \Sigma xP_2 - \Sigma yP_1.$$

(*d*) Combining R_1 and R_2 gives R in *magnitude* and *direction*, but not in position.

$$R = \sqrt{R_1^2 + R_2^2}, \qquad \tan\theta_r = \frac{R_2}{R_1}.$$

(*e*) Combine M and R by **33**, and find the *position* of the resultant *force*, using the equation $p = M/R$. The balancing force is of course the above resultant reversed.

(*f*) The point of application cannot be explicitly found, but the *line* of action can, and any point in that line can be taken as the point of application provided it be a point in or on the body upon which the given forces act.

If x' and y' are the co-ordinates of a point in the action line of R, we have the equation of that line

$$M = x'R_2 - y'R_1$$

whose intercepts on the axes are readily found and the action line can be drawn and the value of p be checked.

This method can be made clear by two examples.

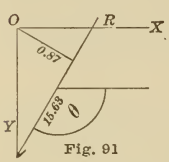


Fig. 91

Ex. 1. Find the resultant of the forces shown in the following table which gives magnitudes, co-ordinates of points of application, and directions of forces. The table also shows how to tabulate the work of calculation. Only two decimals are used in the natural sines and cosines. The forces shown in Fig. 91 are fairly represented in the table.

F	x	y	θ	F_1	F_2	xF_2	yF_1
10	−5	−4	45°	+ 7.07	+ 7.07	−35.35	−28.28
12	0	−6	120°	− 6.00	+10.39	0.00	+36.00
8	2.5	−1	300°	+ 4.00	− 6.93	−17.33	− 4.00
16	2	2	180°	−16.00	0.00	0.00	−32.00
6	4	1	40°	+ 4.62	+3.84	15.36	+ 4.62
				$R_1 = -6.31$	$R_2 = 14.37$	−37.32	−23.66

Results: see Fig. 91

$$R = (R_1^2 + R_2^2)^{\frac{1}{2}} = 15.63 \qquad \theta = \arctan \frac{R_2}{R_1} = 113°40'.$$

$$M = -37.32 + 23.66 = -13.66. \qquad p = M/R = -0.87.$$

Ex. 2. Since both R_1 and R_2 are zero, the resultant is not a single force, but a couple whose moment may be found.

F	x	y	θ
24	3	−1	90°
14	−8	+10	300°
40	4	4	135°
45.5	+6	−6	298°2′

88. III. **Semi-graphic method.** This requires an accurate drawing of the force lines so that in each case its perpendicular distance from some convenient central point O can be measured. If possible, O should be at the intersection of two force lines. Call the perpendiculars p_1, p_2, etc.

(*a*) Imagine that every force F is resolved into an equal parallel force at O, and a couple whose moment is Fp. Measure the perpendiculars and calculate the value of $M = \Sigma Fp$.

(*b*) Beginning at O (or some more convenient point), draw the force polygon of the transferred forces which meet at O, thereby finding the magnitude and direction of the resultant R.

(*c*) Combine the resultant force R with the resultant moment M, as already pointed out in the last method, and find p.

The student should use again the data shown in Fig. 89, and in the Table under the II Method, and so check up all former results.

If in any new problem $M=0$, and $R>$ or <0, the final resultant is a force R acting thru O.

If $R=0$ and $M=0$, the given forces balance and the body acted upon is in equilibrium.

The above three methods of solving problems of co-planar forces should be well mastered. The student will see that each method has at times special advantages.

89. IV. **The funicular, or chain polygon.** *Definition*: A **chain polygon** is a device for finding graphically a point in the resultant of a system of co-planar forces acting at different points on a solid body or frame, by the introduction of a series of *internal forces* which successively balance the given forces, and which are themselves balanced by two forces whose lines of action intersect and thus determine a *point in the resultant required.* The full meaning of this definition will be seen if a solution is closely followed.

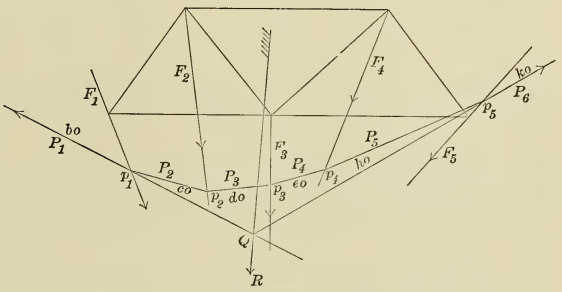


Fig. 92 a

1. Let a trapezoidal frame, consisting of seven bars with pin-joints, be acted upon, as shown in Fig. 92(*a*), by five co-planar forces F_1 . . . . . F_5.

The resultant of the system is to be found.

The magnitudes and directions are shown in (*b*), as in the last section. The open force polygon is *BCDEHK*, and the *Resultant*, *BK*, is found in *magnitude* and *direction*.

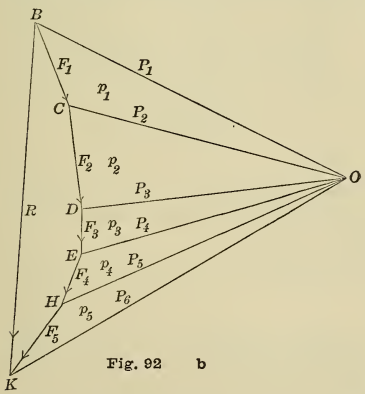


Fig. 92 b

2. We are now to find its position.

From any convenient point O on the right of the force polygon, Fig. 92*b*, draw right lines to *B*, *C*, *D*, *E*, *H* and *K*. These lines represent *tensions* in the links of our proposed

chain. The force, F_1, acts between the two links whose tensions are BO and CO. The force lines of F_1, and the two tensions form a *static triangle* for the pin at p_1, which point is taken at random on the action line of F_1, in Fig. *a*. If parallel to BO, we draw bo so as to intersect the line of action of F_1, and then thru the point of intersection draw co parallel to CO, we have the lines of action of the three forces (viz.: F_1 and the first and the second links of the chain) which balance.

Producing co till it intersects the line of action of F_2, and drawing do parallel to DO, we have the action lines of the forces whose static triangle is COD. In the same way drawing do, eo, fo and ko, we have all the forces balanced except bo and ko. The tensions in bo and ko must balance them all, and this is shown by the static triangle BOK, and hence bo and ko must balance the *resultant*. It follows that the point where they intersect, Q, is in the resultant R. Thru Q, parallel and equal to BK, the resultant of F_1, F_2, F_3, F_4, F_5 can now be drawn.

3. The point O is called the "Pole"; the lines meeting at the pole are called "rays"; Fig. 92 (*a*) is called the "space diagram"; Fig. 92(*b*), is the force diagram; and the polygon in (*a*) formed by the *interior* links and the lines P_1Q and P_5Q, is called the *Equilibrium Polygon*.

90. Different chains. The student can choose a second "Pole" for O, and so have a different chain, and find a different Q, but if the figure is carefully drawn the new point will still be on R.

Great freedom is allowed in drawing a chain polygon just as there is in a force polygon; the forces may be taken in any order, and it is quite possible that one or more links would be in compression, *i. e.*, be struts instead of ties.

1. When the given forces are all vertical and acting down, the force diagram should take the forces in order from *left* to *right*, and the pole should be on the *right*, if the links of the chain are to be in tension. If the pole be taken on the *left*, or if the forces be taken in the *reverse* order, the links will be *struts* and they will form an arch. It is not necessary that the points P_1, P_2, etc., are actually on the body. Since the shape and extent of the body are indefinite, the pins of the imaginary chain may be off or on.

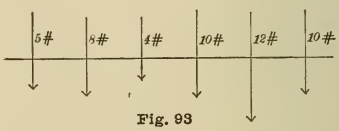


Fig. 93

Examples.

Ex. 1. Find the *chain* (made of tie bars and pins) which will balance the above forces, assuming that the end links are anchored.

Ex. 2. Find an arch which will balance the above forces acting on a body, assuming that the end struts have suitable supports.

2. If the force polygon closes, thereby showing that the resultant is not a single force, the first and last rays coincide, and the first and last links in the chain will represent equal, opposite and parallel forces, thus forming a *balancing* couple. The system shown in Fig. 94 gives a resultant couple whose moment is balanced by $P_1 p$, see Fig. (*a*).

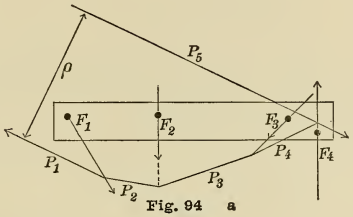


Fig. 94 a

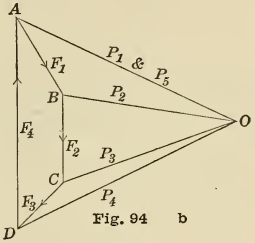


Fig. 94 b

3. Should the lines of action of P_1 and P_5, in Fig. (*a*), coincide, there would be no couple, and as both $R=0$ and $M=0$, and the given forces would be completely self-balanced.

91. The closing link and vertical supports. Having found the resultant of a system of forces by the IV. method, it is often necessary to resolve it into parallel components acting thru given points. This is readily done by the device of inserting an *additional link* connecting the first and last links of the chain, or the first and last bars of an arch. The stress of this new link combined with the old terminal links separately gives the parallel components of the resultant. This method will be illustrated.

Let the given forces be vertical, and let one of the required parallel components, which will, of course, be vertical, pass thru the point A. Fig. 95. Draw the force polygon $BCDK$, with the resultant BK.

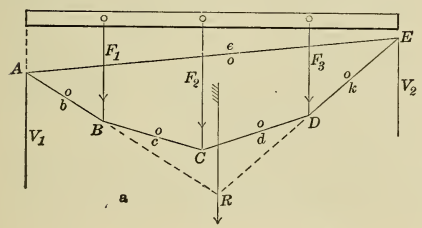
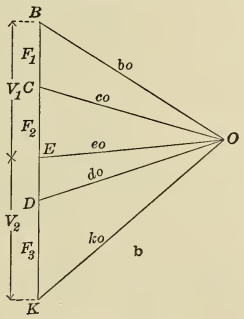


Fig. 95



Choosing now a pole for the stresses in the chain, draw the "rays" and construct the chain polygon as before, starting at A, and find the position of the resultant R.

Now, let us suppose that the other parallel component of R is to pass thru the point E, in the last link of the chain. We then insert a new link from A to E, and draw the "ray" parallel to it from O in the force diagram. The length *eo* gives the stress of this link and the two components of R, which are BE and EK in the stress diagram,

are determined. If these components are individually supported and balanced by direct forces, V_1 and V_2, we have parallel supports which would completely balance a given system of forces.

If now we consider V_1 and V_2 as additional forces acting upon the given body, we have a set of vertical forces which balance, and the complete closed chain forms the Equilibrium Polygon. It is, in fact, the "static polygon" of the chain.

The reader should take note of the static triangles of the several points or pins, A, B, C, D and E, in Fig. 95(b).

92. When both vertical and inclined forces act on a structure, as in the case of vertical loads and wind pressures on a roof, the resultant is in general not vertical. If now one of the supports is to be vertical, the other must be inclined, and the resultant must be resolved into two components, one of which is vertical and acting from the point where the support is to be vertical, the other inclined and acting through the point where the inclined support must act.

In such a case, the line of the resultant already found, should be produced until it intersects the vertical support-line; a line drawn from their intersection to the other support will give the direction of the inclined support.

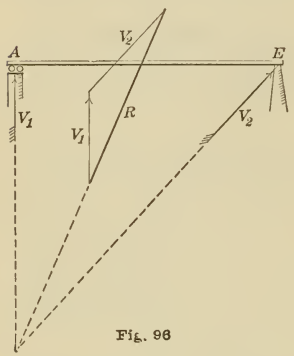


Fig. 96

Suppose the resultant of a system of vertical and inclined forces, acting on a roof, has been found, and is represented by R. Fig. 96. The two supports are at A and E. The roof is anchored at E, and rests on rollers at A, thereby making the support V_1 at A vertical.

The above furnishes a method of finding the supports for the problems solved in Graphical Statics, Chapter XI.

93. If both supports are to be inclined, any point in the line of action of the resultant may be taken as the point from which the two supporting action-lines shall pass, and their inclinations may be chosen at will. In such a case the supports, V_1 and V_2, are readily found.

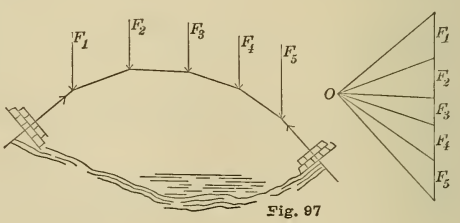


Fig. 97

94. An equilibriated arch. In the case of an equilibriated arch, the supports are always inclined as illustrated below where the given forces are all vertical. Fig. 97.

The student who later makes a special study of **arches** will find another and most important use of the equilibriated arch.

Problem.

Let the student assume a series of equal vertical loads, placed at equal intervals, and support them by an equilibriated *arch*; and again by a *chain*; and see how closely the balanced pins lie in parabolic arcs.

CHAPTER VI.

Forces in Space.

95. Case I. **When three forces meet at a point. Graphical.** When three forces not in the same plane meet at a point, they cannot possibly balance; they can, however, be balanced by a fourth force acting thru the common point. The fourth force must be equal and directly opposite to the resultant of the three. Hence, to find the equilibriating force, we first find the resultant of the given forces. The resultant can be found graphically by drawing in different planes. Thus the resultant of two can be combined with the third in a new plane. The resultant vector of three forces is evidently the diagonal of a parallelopiped.

R_1, Fig. 98, is the resultant of P_1 and P_2; and the resultant of R_1 and P_3 is R, which is seen to be the diagonal of a parallelopiped, which has the given force lines as diverging (or converging) edges. The drawing, however, is not a scale drawing, and has little value beyond giving clear mental concepts. When four forces balance, we have a static polygon (putting $P_4 = -R$), $OABC-O$, which is a warped static polygon.

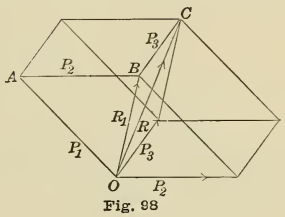


Fig. 98

96. **Analytic.** 1. The only convenient method of recording the data and reaching exact solutions of such space problems is the analytic, using three rectangular axes, and specifying directions by the angles which force lines make with them separately. We shall, in general, assume that OX and OY are horizontal, and that OZ is vertical. The logic of the arrangement has already been explained.

2. The angles which a force P, makes with the axes OX, OY and OZ, are respectively α, β, γ, the arrows always pointing *from* the origin.

It is evident that these angles are not wholly independent of each other. The angles lie in different planes, no one of which (except in special cases) is a co-ordinate plane.

Suppose we know, at first, only that α is 40°. At once we picture a cone of revolution with O for its vertex, OX for its axis, and a half-vertical angle of 40°. If now we learn that β is 60°, we think of another cone mounted on OY with a half vertical angle of 60°. Of course, these cones intersect in two elements, one above and one below the plane of XY. There are therefore two possible values of γ which are supplements of each other, which are both determined, and P is either in the *first* or *fifth* triedral angle.

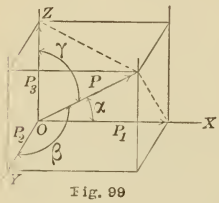


Fig. 99

3. The trigonometric equation showing the relation of the three angles is found as follows:

If the force P be resolved into its rectangular components P_1 along OX, P_2 along OY, and P_3 along OZ, we have by inspecting Fig. 99:

$$P_1 = P\cos\alpha$$
$$P_2 = P\cos\beta$$
$$P_3 = P\cos\gamma$$

Squaring and adding, and taking note of the fact that

$$P_1^2 + P_2^2 + P_3^2 = P^2, \text{ we have}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1; \tag{1}$$

hence, when

$$\alpha = 40° \text{ and } \beta = 60°$$

$$\cos\gamma = \pm\sqrt{(1 - \cos^2 40° - \cos^2 60°)}$$

giving the two values of γ already noted.

4. When the resultant of several forces acting at the same point in space is desired, for the purpose of finding the balancing force, proceed as follows:

(*a*) Assume the common point of action as the origin, and if possible, take the co-ordinate planes so that at least one of the forces will lie in a co-ordinate plane.

(*b*) Ascertain the direction angles of every given force.

(*c*) Resolve every given force into its three rectangular components.

(*d*) Add the components on the separate axes, getting

$$\left.\begin{aligned} R_1 &= \Sigma P\cos\alpha \\ R_2 &= \Sigma P\cos\beta \\ R_3 &= \Sigma P\cos\gamma \end{aligned}\right\} \tag{2}$$

(*e*) Find R by the equation

$$R = \sqrt{R_1^2 + R_2^2 + R_3^2}$$

and α_r, β_r and γ_r by the equations

$$\begin{aligned} \cos\alpha_r &= R_1/R \\ \cos\beta_r &= R_2/R \\ \cos\gamma_r &= R_3/R \end{aligned} \tag{3}$$

97. Illustrative example. A weight of 100 lbs. is to be suspended to a small ring directly over it, and that ring is to be held in position by *three* cords. The positions and tensions of two of the cords are known, but nothing is known of the third cord and force except that it must balance the three others. The unknown force is to be fully determined.

The positions and magnitudes of the given forces are shown in Fig. 100.

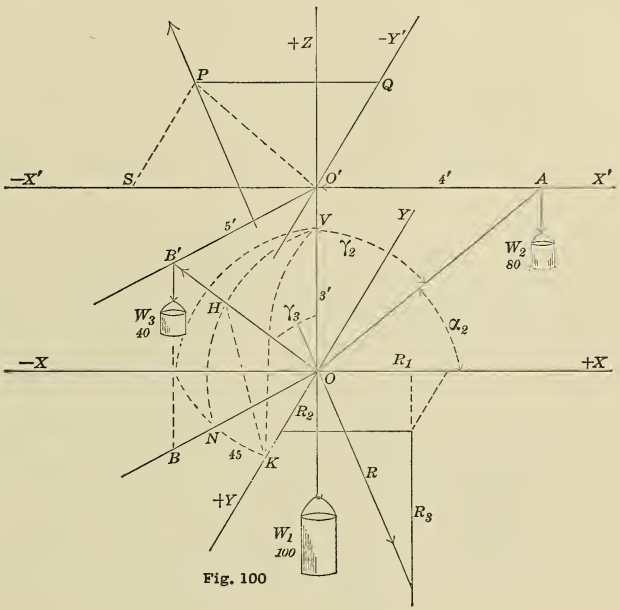


Fig. 100

Solution. Take the ring as the origin, and the plane of W_1 and W_2 as the plane XZ. A horizontal gridiron frame, three feet (3′) above the ring, supports light pulleys, over which cords are to run from the weights $W_2 = 80$ lbs., and from $W_3 = 40$ lbs. The tension line of W_2 is thru A, which is in the plane $X'Z$, and 4′ from O'; hence

$$\cos\alpha_2 = \frac{4}{5};\ \cos\gamma_2 = \frac{3}{5};\ \cos\beta_2 = 0$$

The tension line of $W_3 = 40$ lbs. is in a vertical plane which *bisects* the angle $(+Y)O(-X)$, and the pulley B' is 5′ from O', and 3′ above the plane XY; hence $\tan\gamma_3 = \frac{5}{3}$, and $\cos\gamma_3 = \frac{3}{\sqrt{34}}$.

To find β_3, assume a spherical surface, with radius unity and center at O, cutting OY at K, and the force line of W_3 at H.

The bisecting plane, passing thru B' and OZ, cuts the sphere in the arc HN, which is the complement of γ_3. Drawing the arc NK cut from the unit sphere by the plane XY, we have a spherical right triangle HNK, right-angled at N. Its legs are $KN = 45°$, $HN = 90 - \gamma_3$, while $HK = \beta_3$.

Hence, by Napier's Rules,
$$\cos\beta_3 = \cos 45° \sin\gamma_3 = \frac{5}{34}\sqrt{17}.$$

and by **96** (1)
$$\cos\alpha_3 = -\sqrt{1 - \frac{9}{34} - \frac{25}{68}} = -\frac{5}{34}\sqrt{17}.$$

The reason for the negative value of $\cos\alpha_3$ is evident.

Table giving *data* and *solution*:

P	$\cos\alpha$	$\cos\beta$	$\cos\gamma$	$P\cos\alpha$	$P\cos\beta$	$P\cos\gamma$
$W_1 = 100$	0	0	-1	0	0	-100
$W_2 = 80$	0.800	0	0.600	64.00	0	48.00
$W_3 = 40$	-0.606	$+0.606$	0.515	-24.24	$+24.24$	20.60
$P = ?$	α_r	β_r	γ_r	$R_1 = 39.76$	$R_2 = 24.24$	$R_3 = -31.40$

$$R = \sqrt{R_1^2 + R_2^2 + R_3^2} = 56.16$$

The position of R is seen from its three components, R_1, R_2 and R_3, in Fig. 100, to be in the 5th triedral angle.

The angles α_r, β_r, γ_r are found from their cosines, which are
$$\cos\alpha_r = \frac{39.76}{56.16},\ \cos\beta_r = \frac{24.24}{56.16},\ \cos\gamma_r = \frac{-31.4}{56.16}$$

If we represent the direction angles of the *balancing force* by
$$\alpha' = 180° - \alpha_r,$$
$$\beta' = 180° - \beta_r,$$
$$\gamma' = 180° - \gamma_r,$$

we have
$$\cos\alpha' = -\cos\alpha_r = -\frac{39.76}{56.16}$$
$$\cos\beta' = -\cos\beta_r = -\frac{24.24}{56.16}$$
$$\cos\gamma' = -\cos\gamma_r = +\frac{31.4}{56.16}$$

and the Balancing Force is completely found.

To exhibit the position of its line of action more clearly, it can be said that the force required to balance the three given forces *acts up into the third triedral angle.* It pierces the plane $X'O'Y'$ (which is three feet above XOY) at a point P, which is:

$3\tan\gamma'$ distant from O',
$3\sec\gamma'\cos\alpha'$ distant from $O'Y'$,
$3\sec\gamma'\cos\beta'$ distant from $O'X'$.

F	α	β	γ
20	60°	$<90°$	50°
85	$>90°$	30°	90°
100	135	100°	>90
60	30°	120°	?
$R=?$	$\alpha_r=?$	$\beta_r=?$	$\gamma_r=?$

Problem.

Let the student mentally picture the concrete situation whose data are given in this table, and represent them *graphically*; then derive the answers to the five questions. One square in the table is purposely left vacant.

Queries.

1. What does it mean when one of the components of R is zero?
2. What does it mean when two of the components of R are zero?
3. What does it mean when all three components of R are zero.

98. Case II. **When the forces do not converge to a point, but act at different points and in different directions.** The General Case. The analysis is an extension of the analysis for non-concurrent forces in a plane. A point of application is given by its three co-ordinates, x, y and z. The direction of a force by three direction angles, α, β, γ, with reference to three rectangular lines (parallel to the co-ordinate axes) drawn thru the point of application. The steps in the process of finding a resultant are as follows:

(*a*) *Resolve every force into its three components*:

$$P_1 = P\cos\alpha$$
$$P_2 = P\cos\beta$$
$$P_3 = P\cos\gamma$$

(*b*) *Resolve every component into an equal parallel force at O, and two moments,* with reference to the two co-ordinate axes perpendicular to it. This must be illustrated. Let P_1, Fig. 101, the component acting at the point $A(x, y, z)$ be resolved into an equal force, P_1', acting B, and a couple zP_1 (*right-handed*) about the Y axis. Next think of P_1' as acting at C, and resolve it into an equal parallel force, P_1'', at O, and a *left-handed* couple yP_1 about the axis Z. Of course P_1 produces no moment about the X axis, with which it is parallel.

We thus get for P_1 acting in space:

a force P_1 at O, along OX;
a moment about OY of $+zP_1$;
and a moment about OZ of $-yP_1$.

In like manner, for the component P_2, we shall get:

a force P_2 at O, along OY;
a moment about OZ of $+xP_2$;
and a moment about OX of $-zP_2$.

And for the component P_3:

a force P_3 at O, along OZ;
a moment about OX of $+yP_3$;
and a moment about OY of $-xP_3$.

Summing these results, we get for every force P in space:
Three components, P_1, P_2, P_3, at the origin of co-ordinates;

A moment about OX of $yP_3 - zP_2 = M_x$
A moment about OY of $zP_1 - xP_3 = M_y$.
A moment about OZ of $xP_2 - yP_1 = M_z$.

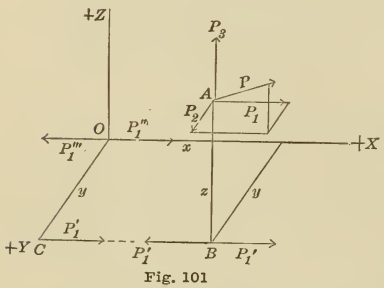


Fig. 101

A careful examination of Fig. 101 will make all these moments intelligible.

If the results are tabulated the student will notice the sequence of letters and numbers.

P	M_x	M_y	M_z
$P_1P_2P_3$	$yP_3 - zP_2$	$zP_1 - xP_3$	$xP_2 - yP_1$

Note the sequence of letters and numbers, in Fig. 102.

The student will also notice that the moment about OZ or M_3 is identical with that found for the axis OZ, when the force was in the plane XY, and there could be no moment about OX and OY as both z and P_3 were zero.

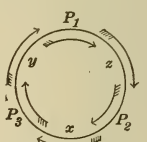


Fig 102

(*c*) After all the forces are resolved as above, sum the results.

$$\left.\begin{array}{ll} \text{Find } \Sigma(P_1) = R_1 & \Sigma(yP_3) - \Sigma(zP_2) = M_1 \\ \text{Find } \Sigma(P_2) = R_2 & \text{and } \Sigma(zP_1) - \Sigma(xP_3) = M_2 \\ \text{Find } \Sigma(P_3) = R_3 & \Sigma(xP_2) - \Sigma(yP_1) = M_3 \end{array}\right\} \quad (4)$$

(*d*)
$$\left.\begin{array}{l} R = \sqrt{(R_1^2 + R_2^2 + R_3^2)} \\ M = \sqrt{(M_1^2 + M_2^2 + M_3^2)} \end{array}\right\} \quad (5)$$

This combination of three moments about three separate rectangular axes is in accord with what was shown in (II).

In general, the resultant of such a system of forces in space is a *resultant force* acting at the origin, *and* a *resultant moment* about an axis in space.

(*e*) The direction angles for R are found by the equations

$$\left.\begin{aligned}\cos\alpha_r &= R_1/R\\ \cos\beta_r &= R_2/R\\ \cos\gamma_r &= R_3/R\end{aligned}\right\} \tag{6}$$

And the direction angles of the axis of the resultant moment are found by the equations

$$\begin{aligned}\cos\lambda &= M_1/M\\ \cos\mu &= M_2/M\\ \cos\nu &= M_3/M\end{aligned} \tag{7}$$

2. The axis of M may be assumed to pass thru O. If the axis of M is parallel to, or coincides with, R, no further combination is possible.

If the axis of M and R make an angle θ, M may be resolved into two new components, $M\cos\theta$, and $M\sin\theta$. The component, $M\sin\theta$, can be combined with R, thereby shifting R to a parallel position distant from O by a perpendicular L.

$$L = \frac{M\sin\theta}{R}$$

The perpendicular is drawn normal to a plane which passes thru R and is parallel to the axis of M. The resultant of the system is then R in this new position and the moment $M\cos\theta$. If θ is a right angle, $M\cos\theta$ is zero, and the resultant is a single force.

If $M=0$, the resultant is a force R at the origin of co-ordinates. If $R=0$ while M is not zero, the resultant is a couple with moment M.

If $R=0$ and $M=0$, the system is self-balanced and the body acted upon is in equilibrium.

While an engineer rarely has occasion to solve a general problem in statics, it is well to embody the method in the solution of a problem with assumed data.

Problem.

Solve the problem in the table, filling out blank spaces systematically.

P	x	y	z	α	β	γ	P_1	P_2	P_3	yP_3	zP_2	zP_1	xP_3	xP_2	yP_1
8	4	5	12	75	50	<90°									
16	0	0	0	30		60									
10	−2	+3	−4	135°	60	>90									
20	8	−1	−6	>90	70°	45°									
							R_1=?	R_2=?	R_3=?						
			R=?		M=?					M_1=?		M_2=?		M_3=?	

The student will notice, when $P=8$, that $\cos\gamma = +\sqrt{\sin^2\alpha - \cos^2\beta}$, since γ must be acute.

The actual value of γ need not be found, as only its cosine is wanted.

The value of β for the second force the student must supply himself.

The above system of forces can be balanced only by balancing R, and M, either by a single force or a combination of a single force and a single couple.

CHAPTER VII.

Centroids and Centers of Pressure.

99. 1. The two most important applications of the theory of parallel forces are found in problems which require the concentration of distributed forces. In Chapter I it was assumed that such concentration was possible, and we did not hesitate to give the total magnitude of a force distributed over a surface or thru a volume; but we did not attempt to determine the exact location of the line of action of the concentrated force. This chapter will be devoted to finding the positions of concentrated or resultant forces when the law of distribution is definitely known.

100. The point in a surface where the resultant of a distributed force pierces that surface, is called the "Center of Action."

As has been said, all of our problems are more or less ideal, but they closely approximate the real. We assume, for example, that the steam or air pressure upon the vertical face of a piston is uniform, tho we know that such is not absolutely the case. We assume that the pressure between the corner stone of a building and the material below it is uniformly distributed, when it probably is not uniform. We assume that the earth's attraction upon a cubic inch of material in a block of wood, stone or metal, is uniform thruout the block, tho we have no idea that such is the absolute truth.

101. The phrase: "The intensity of the surface action at a point" means: the amount of action or force there would be on a unit of surface if thruout the unit surface the force were uniformly what it is upon the infinitesimal area surrounding the given point.

Since the *magnitude* and *direction* of a *uniformly distributed force* cannot affect the position of the Center of Action, we may assume that the direction is normal, and that the intensity is p, which is as small as one likes.

102. Centroids of plane surfaces. The point where the *resultant of a uniformly distributed force* pierces a *plane surface* is called the "*Centroid*"* of that surface.

In some simple geometrical surfaces the centroid is evident by symmetry, without demonstration. For example: The circle, the ellipse, a rectangle, all regular polygons. If a surface has an axis of symmetry, the centroid is on that axis; two such axes determine the centroid. If a surface can be divided into parts such that the centroid of every part is known, and known to lie on a certain straight line, the centroid of the surface is known to lie on that line. The number of parts may be finite or infinite. Hence the centroid of a triangle is at the intersection of its medial lines.

103. When a surface of action can be divided into finite parts whose several areas and centroids are known, the problem of finding the centroid of the entire surface falls directly under parallel forces in space. Chapter III.

It was proved in geometry that the point which we now call the centroid of a plane triangle is on a medial line two-thirds of its length from the vertex end. **The centroid of a plane trapezoid** is found graphically as shown in Fig. 103. The diagonal AB divides the surface into two triangles whose centroids determine the right line, ab. The diagonal CD, gives two new triangles, and a second line, cd, which, with ab determines G, the centroid of the trapezoid.

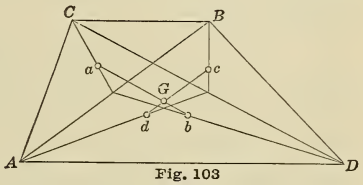


Fig. 103

The centroid of any rectilinear plane surface is found by an extension of the method just used.

104. The general case. When the outlines of a surface are curved and irregular, and there are no axes of symmetry, the *Centroid* must be found by the doctrine of moments and the use of the Calculus.

1. Let the uniform force intensity be p. If A represents the whole area of the surface of action, ΔA will represent a portion of that surface, and $p\Delta A$ will represent the magnitude of the force on that portion. As the whole is equal to the sum of its parts

$$R = \Sigma(p\Delta A)$$

* The author prefers the word "Centroid" to the phrase "Center of Gravity" when speaking of plane areas, yet the latter is in very common use.

The introduction of thin flat plates of heavy material fails to hide the absurdity of speaking of the *gravity of a surface;* and to go further and assume that the plate is half above and half below the given area in order to bring the C. G. of the plate into coincidence with the Centroid of the surface is confusing to a novice.

In this case the summation must be made by the Calculus. Instead of the sum of a finite number of finite terms, we must have the sum of an infinite number of infinitesimal terms. Hence we shall use $\int$ for Σ, and d for Δ, and the above formula becomes

$$R = \int p dA$$

It is here and elsewhere assumed that the student is familiar with the ordinary methods of integration, but he may be less familiar with their application to practical problems. Accordingly, some fundamental matters will be fully explained. The expression dA is the numerical measure of a portion of the surface A, and preferably represents the product of a length by a breadth. Both factors may be infinitesimals, as $dxdy$; or one may be finite, as ydx, or $(y_2-y_1)dx$; see Figs. 104 and 105. If $dA=dxdy$, two integrations will be required; if $dA=ydx$, only one integration is necessary; hence the second form is preferred, when it can be used.

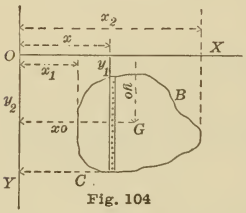


Fig. 104

2. Let BC be a surface bounded by outlines not at present defined. General formulas for the position of its centroid are to be found. Let the surface A be divided into elements parallel to the axis OY, with a constant width, dx. The elementary area of a single strip is $(y_2-y_1)dx$, according to the lettered Fig. 104. If p be the intensity of the distributed force, the force acting upon the strip is $(y_2-y_1)pdx$. The moment of this force about OY is $x(y_2-y_1)pdx$, and the sum of the moments of all the elements is

$$M_y = \int_{x_1}^{x_2} x(y_2-y_1)pdx = p\int_{x_1}^{x_2}(y_2-y_1)xdx \qquad (1)$$

since p is constant during the integration.

Now, the total force acting on the surface, BC, is $p\int_{x_1}^{x_2}(y_2-y_1)dx$ which is p times the area, or pA. When this force is centralized it will act at the centroid of the surface whose co-ordinates we will call x_o and y_o. Then the moment of the centralized or resultant force about OY will be $x_o p\int_{x_1}^{x_2}(y_2-y_1)dx = M_y$

Now follows an *Axiom of Mechanics*:

The moment of the whole is equal to the whole of the moments.

$$x_o p\int_{x_1}^{x_2}(y_2-y_1)dx = p\int_{x_1}^{x_2}(y_2-y_1)xdx$$

$$x_o R = M_y$$

Hence
$$x_o = \frac{\int_{x_1}^{x_2}(y_2-y_1)x\,dx}{\int_{x_1}^{x_2}(y_2-y_1)\,dx} = \frac{M_y}{R} \qquad (2)$$

3. **The method of integration** is determined in every case by the equations of the boundary lines of the surface.

The disappearance of p shows that x_o depends not on the absolute value of p, but upon its *uniformity*. We will therefore assume that $p=1$, and numerically $R=A$.

4. The co-ordinate, y_o, is found in a similar manner. If the same elementary strip, $(y_2-y_1)dx$, is used, we must get its moment about OX by multiplying its area by the ordinate to its central point, which is $\frac{y_1+y_2}{2}$. Hence our value of y_o will appear

$$y_o = \frac{\frac{1}{2}\int_{x_1}^{x_2}(y_2^2-y_1^2)\,dx}{\int_{x_1}^{x_2}(y_2-y_1)\,dx} = \frac{M_x}{R} = \frac{M_x}{A}. \qquad (3)$$

5. If a new element is taken parallel to the axis of X, the value of y_o appears thus:—

$$y_o = \frac{\int_{y_1}^{y_2}(x_2-x_1)y\,dy}{\int_{y_1}^{y_2}(x_2-x_1)\,dy} = \frac{M_x}{A} \qquad (4)$$

There are other general methods depending upon the form of the surface element and upon the use of circular or polar co-ordinates. A variety of illustrative examples is chosen so as to make every method clear.

105. Examples illustrating methods of finding centroids.

Ex. 1. The centroid of a parabolic half segment. Fig. 105.

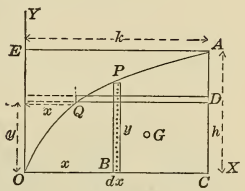


Fig. 105

Let the surface be OAC, and the element be $PB=ydx$. The Equation of the curve is $y^2=ax$. N. B.—While we are to find x_o and y_o, the co-ordinates of the *interior* point G, the student must bear in mind that x and y thruout our analysis are the co-ordinates of points P, on the *bounding curve*.

Since $p=1$, the force on our element is ydx, and its moment about the axis OY is

$$dM_y = xydx$$

Hence
$$M_y = \int_o^k a^{\frac{1}{2}}x^{\frac{3}{2}}dx = \frac{2}{5}\,k^2h$$

It was found in the calculus that

$$\text{Area of } OAC = \frac{2}{3}\,kh = A = R.$$

Hence by **104**, $$x_o = \frac{M_y}{A} = \frac{3}{5}\,k.$$

The centroid of the strip element, BP, is at its middle point $y/2$ from OX; hence, the moment of the element about OX is

$$dM_x = \frac{y}{2}\,ydx$$

and $$M_x = \frac{1}{2}\int_o^k y^2dx = \frac{a}{2}\int_o^k xdx = \frac{ak^2}{4}.$$

$$y_o = \frac{M_x}{A} = \frac{3}{8}\,h.$$

106. The student must not be wedded to one form of a surface element. The element $QD = (k-x)dy$, would have served equally well. We see at once using subscripts 1 and 2 in place of x and y:

$$dM_1 = (k-x)ydy$$

$$M_1 = \int_o^h kydy - \int_o^h xydy$$

$$= \frac{kh^2}{2} - \frac{h^4}{4a} = \frac{kh^2}{4}$$

$$y_o = \frac{M_1}{A} = \frac{3}{8}\,h$$

$$dM_2 = (k-x)\,\frac{k+x}{2}\cdot dy$$

$$M_2 = \frac{1}{2}\int_o^h (k^2-x^2)dy = \frac{hk^2}{2} - \frac{h^5}{10a^2} = \frac{2}{5}\,k^2h$$

$$x_o = \frac{M_2}{A} = \frac{3}{5}\,k.$$

107. The algebraic method. This method is based upon a full set of data, and is best illustrated by an example. Suppose we wish to know the centroid of the parabolic spandril $OQAE$, knowing the centroid of the rectangle $OCAE$, and the centroid of the half segment OCA, as found above.

The three centroids are on the same straight line. Why? And the centroid of the spandril is *twice* as far from the center of the rectangle as is the G of the segment. Why?

The Axiom of Moments with respect to each axis, OX and OY, serves our purpose. Letting x_o, y_o be the unknown co-ordinates of the centroid of the spandril, and using the data of the rectangle and segment, we have the two moment equations.

$$M_2 = hk \times \frac{1}{2}k = \frac{2}{3}hk \times \frac{3}{5}k + \frac{1}{3}hk \times x_o$$

$$M_1 = hk \times \frac{1}{2}h = \frac{2}{3}hk \times \frac{3}{8}h + \frac{1}{3}hk \times y_o$$

whence

$$x_o = \frac{3}{10}k$$

$$y_o = \frac{3}{4}h.$$

This indirect method is often very convenient.

Ex. 1. Find the Centroid of the above Spandril by direct integration.

Ex. 2. Find the Centroid of a quadrant of an ellipse.

Ex. 3. In Fig. 103, let the non-parallel sides, AC and DB, be produced till they meet in O. Find graphically the centroid of the triangles OAD and OCB, and measure their distance from O. Then by the algebraic method find the distance to the centroid of the trapezoid.

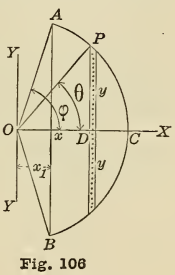


Fig. 106

108. Find the centroid of a **circular segment**, introducing circular co-ordinates.

Solution. The segment is ABC, Fig. 106. Since OX is an axis of symmetry, the centroid is on that axis and $y_o = 0$, The element is $2ydx$. From the figure

$$x = r\cos\theta,\ y = r\sin\theta,\ dx = -r\sin\theta\, d\theta,\ x_1 = r\cos\phi.$$

$$M_y = \int_{x_1}^{r} 2yx\,dx = -2\int_{\varphi}^{o} r^3\sin^2\theta\cos\theta\, d\theta = 2r^3\int_{o}^{\varphi}\sin^2\theta\cos\theta\, d\theta, = \frac{2}{3}r^3\sin^3\phi.$$

$$A = 2\int_{x_1}^{r} y\,dx = -2r^2\int_{\varphi}^{o}\sin^2\theta\, d\theta = 2r^2\int_{o}^{\varphi}\sin^2\theta\, d\theta = r^2(\phi - \sin\phi\cos\phi).$$

The student will take note that the expression for A (the area of the segment) is explicitly the area of the sector $OACB$ *minus* the area of the triangle OAB: that is

$$A = r^2\phi - r^2\sin\phi\cos\phi.$$

Dividing M_y by A we have

$$x_o = \frac{M_y}{A} = \frac{2}{3}\cdot\frac{\sin^3\phi}{\phi - \sin\phi\cos\phi}\cdot r$$

For a *semi-circle* $\phi = \pi/2$ and

$$x_o = \frac{4r}{3\pi}.$$

NOTE. The student must not be surprised that $dx = -r\cos\theta\, d\theta$ with the negative sign. If he looks at Fig. 106, he will see that as x *increases*, θ *decreases*; hence, if dx is positive, $d\theta$ must be negative, and $(-d\theta)$ is positive.

109. The centroid of a circular sector. The surface element may best be defined by polar co-ordinates. Let the given surface be the sector of a circle, $OABC$, Fig. 107. As OB is an axis of symmetry, we have only to find the distance x_o. This could be found by algebra if we knew the position of the centroid of the circular segment ACB, as well as of the triangle OAC. But the direct method is simpler, as follows:

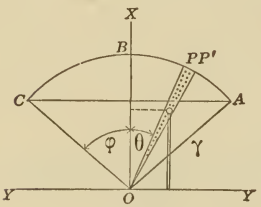


Fig. 107

Let the radius be r, and the half-angle $AOB = \phi$. Draw two consecutive radii in a general position defined by the arcs θ and $d\theta$; the elementary surface is an infinitesimal triangle whose area is $\frac{1}{2}r^2 d\theta$; its centroid is at a distance $\frac{2}{3}r$ from O, and its distance from YY is $\frac{2}{3}r\cos\theta$. Hence, taking moments about YY, and integrating, we have

$$x_o = \frac{2\int_o^{\varphi} \frac{r^2}{2} \cdot \frac{2}{3} r\cos\theta\, d\theta}{2\int_o^{\varphi} \frac{r^2}{2} d\theta} = \frac{2}{3} \cdot \frac{\sin\phi}{\phi} \cdot r$$

Both integrals are doubled so as to include the whole sector.

Corollary.

If $\theta = \pi/2$, the sector is a semicircle and $x_o = 4r/3\pi$ as found in **108.**

Ex. Find the centroid of the "spandril" to a segmental arc.

110. DOUBLE INTEGRATION. **To find the centroid of a sector of a circular ring area,** $EABCDN$. Fig. 108. We now take an elementary part of such a sector enclosed by two cosecutive arcs, and two consecutive radii. This element being an infinitesimal of the second order is treated as a rectangle with the area $(\rho d\theta)\,(d\rho)$. Its moment about YY is $\rho^2 d\rho\cos\theta d\theta$.

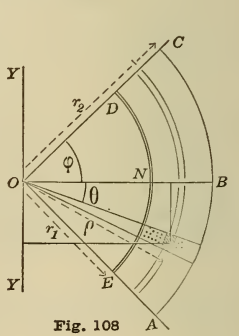


Fig. 108

Hence
$$x_o = \frac{2\int_{r_1}^{r_2}\int_o^{\varphi} \rho^2 d\rho \cos\theta d\theta}{2\int_{r_1}^{r_2}\int_o^{\varphi} \rho d\rho d\theta}$$

The order of integration is immaterial. If we integrate for θ first, we shall get the moment of a sector of an elementary ring as shown by adjacent arcs; its length is $2\rho\phi$ and its width $d\rho$. The next integration would sweep the ring outward from $\rho = r_1$ to $\rho = r_2$.

Had we integrated first for ρ, we should have had a trapezoid lying between two consecutive radii and extending from r_1 to r_2.

Taking the first order, integrating for θ while ρ and $d\rho$ are constant, we get (canceling the 2)

$$x_o = \frac{\sin\phi \int_{r_1}^{r_2} \rho^2 d\rho}{\phi \int_{r_1}^{r_2} \rho d\rho} = \frac{2}{3} \cdot \frac{\sin\phi}{\phi} \cdot \frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$$

This problem could have been done by the algebraic method and **109.**

If $r_1 = 0$, we get the centroid of a sector as in the last example.

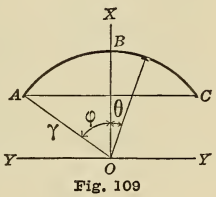


Fig. 109

111. The centroid of a circular arc. Fig. 109. The arc may be supposed to have a breadth of dr, the radius being r. As above, the element is $rdrd\theta$; its moment* about YY is $(rdrd\theta)\ (r\cos\theta)$.

$$M_y = 2r^2 dr \int_o^{\varphi} \cos\theta\, d\theta = 2r^2 \sin\phi dr$$

The area of the arc, or infinitely narrow ring is $2r\phi dr$.

$$\therefore x_o = \frac{M_y}{A} = \frac{2r^2\sin\phi}{2r\phi} = \frac{r\sin\phi}{\phi} \qquad (10)$$

Another form of this result is easily remembered: Since the chord $AC = 2r\sin\phi$, and the arc $= 2r\phi$, we have

$$\frac{x_o}{r} = \frac{2r\sin\phi}{2r\phi} = \frac{\text{chord}}{\text{arc}}$$

which proportion is readily put into words.

* Since we have obliterated the force, p, per square unit of surface, by making it *unity*, we naturally drop into the habit of speaking of a *surface* as tho it were a normal force. The "habit" is quite conventional.

Problem. What is the length of the arc whose centroid is at the center of its central radius?

112. The centroid of a cycloidal area. Suppose the area under consideration is bounded by a cycloidal arc and its base. Fig. 110. It is evident that BH, the longest ordinate, is an axis of symmetry: hence, $x_o = \pi r$, and only y_o is to be found.

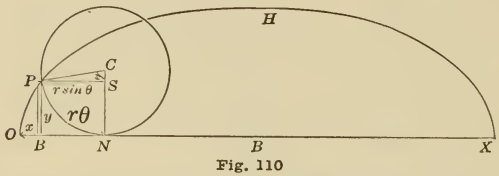


Fig. 110

From the figure, since $ON = r\theta$

$$x = r\theta - r\sin\theta$$
$$y = r - r\cos\theta$$

The element of the surface is ydx, and its centroid is distant from OX a length $y/2$. Hence

$$y_o = \frac{\int_o^{2\pi r} ydx \,.\, y/2}{\int_o^{2\pi r} ydx} = \frac{M_1}{A}$$

From $x = r\theta - r\sin\theta$, we have

$$dx = r(1-\cos\theta)\, d\theta, \text{ so that}$$

$$y_o = \frac{\frac{r^3}{2}\int_o^{2\pi} (1-\cos\theta)^3\, d\theta}{r^2 \int_o^{2\pi} (1-\cos\theta)^2\, d\theta} = \frac{\frac{5}{2}\pi r^3}{3\pi r^2} = \frac{5}{6} r$$

The limits for θ follow from the facts that when $x = 0$, $\theta = 0$; and when $x = 2\pi r$, $\theta = 2\pi$. The uncanceled denominator incidentally shows that the area of the cycloid is just three times the area of the rolling circle.

CENTERS OF ACTION OF DISTRIBUTED FORCES WHICH VARY IN INTENSITY.

113. When a surface action is **a distributed force which varies in intensity** according to a definite law, the center of the force for the whole can be found if the centers of the parts can be found. The most important case is that of a *uniformly varying force*, acting on a plane surface. The following are cases where the variation is approximately uniform:

1. The pressure of water upon the wall of a tank.

2. The normal force acting upon the cross-section of an elastic beam which is slightly bent.

3. The pressure upon the top of a column which has a rigid cap supporting an eccentric load.

4. The pressure on the material under a foundation stone which is eccentrically loaded.

All of these cases will be discussed with more or less fullness.

114. The center of hydrostatic pressure. The *intensity* of internal pressure at any point in any liquid which is at rest in an open tank or vessel is equal to the weight of a column of liquid and the air above it, having for its base a unit area at the point, and reaching to the top of the atmosphere. The weight of such a column of air, with its base at sea level, is about 2,116 lbs per square foot. This produces a pressure of 2,116 lbs. upon the surface of the level liquid, and by so much increases the internal pressure of the liquid at every point. The weight of a column of the liquid itself from the point under consideration to the surface must be added to get the intensity required. See Chapter under Buoyancy.

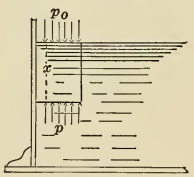


Fig. 111

If p be the required intensity at a depth of x below the surface of a liquid, let w be the *specific weight*, or the weight of a unit of volume, of the liquid, and p_o the atmosphere pressure on the surface of the water; then

$$p = p_o + wx$$

In point of fact, in problems in Applied Mechanics, the pressure p_o is more often balanced by an equal pressure on the other side of the wall or vessel, so that the pressure of still air can generally be left out of consideration. The increased pressure due to moving air, and the diminished pressure caused by rarefaction, require special examination. Accordingly, the *hydrostatic pressure varies with the depth.*

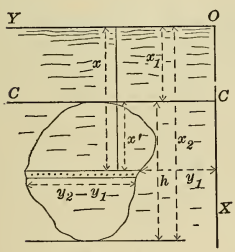


Fig. 112

115. To find the total pressure on a given vertical plane surface and the "center of pressure," we proceed as follows, assuming that the atmospheric pressure is self-balanced:—

1. Take an element of the area of action, at a depth of x below the top of the liquid. Fig. 113. The intensity being the same at all points at that depth, let the element extend across the given surface horizontally, so that its area shall be $(y_2 - y_1)dx$. The pressure at that

depth is xw per unit of surface, and the pressure on the element is $wx(y_2-y_1)dx$, and the total pressure on the entire area is

$$P=w\int_{x_1}^{x_2}(y_2-y_1)xdx. \tag{1}$$

2. The Center of Pressure involves the moment. Take for the moment axis the line CC_1 Fig. 112, tangent to the upper edge of the surface of action. Let $x'=x-x_1$ and $dx=dx'$. The moment of the pressure upon the element is

$$dM_c=w(y_2-y_1)\ (x_1+x')x'dx'$$

and

$$M_c=w\int_o^h(y_2-y_1)\ (x_1x'+x'^2)dx'$$

and

$$x_o=\frac{M_c}{P} \tag{2}$$

The general formula for y_0 is found, when needed, in a similar manner.

116. Let the surface of action A be a rectangle with vertical edges, so that y is constant, and equals b, Fig. 113, so that the area is $A=bh$.

1. Applying **115** (1), we have

$$P=bw.\ \frac{x_2^2-x_1^2}{2}=bwh\ \frac{x_2+x_1}{2}=wA\ \frac{x_2+x_1}{2}=wA\ \frac{h+2x_1}{2}$$

i. e., the *total pressure* on the rectangle $BCDE$ is equal to the area of the rectangle multiplied by the pressure at the "mean depth." This result was almost self-evident, and the same formula must be true of a circle, viz.: the area of the circle times the intensity of the pressure at the center.

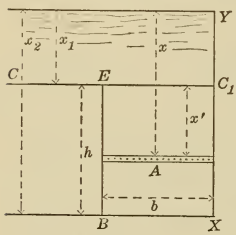


Fig. 113

2. *The moment* about CC_1 of the liquid pressure on the elementary area bdx is $bwxdx\times x'$, as shown in Fig. 113. Since $x=x_1+x'$, and $dx=dx'$, we have for the whole moment

$$M_c=bw\int_o(x_1x'+x'^2)dx'=bw\left(\frac{x_1h^2}{2}+\frac{h^3}{3}\right)$$

$$=\frac{bwh^2}{6}(2h+3x_1)=\frac{wAh}{6}(2h+3x_1)$$

$$\text{and } x_o=\frac{M_c}{P}=\frac{h}{3}\cdot\frac{2h+3x_1}{h+2x_1}$$

in which x_o is measured from the top of the rectangle, CC_1 and x_1 is the distance from CC_1 to the top of the liquid.

Corollary. If the area of action extends to the top of the liquid, $x_1 = o$ and we have

$$x_o = \frac{2}{3}h.$$

117. 1. **Let the immersed area be the triangle** *BBD.* See Fig. 114. With the lettering shown, we have

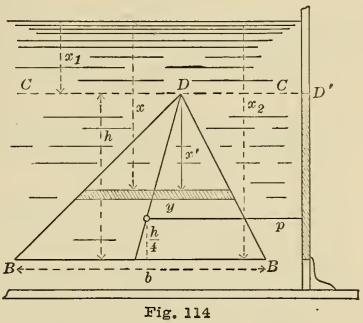


Fig. 114

$$P = w\int_{x_1}^{x_2} xydx \text{ in which}$$

$$y = \frac{bx'}{h},\ x = x_1 + x',\text{ and } dx = dx';$$

Hence $$P = \frac{bw}{h}\int_o^h (x_1x' + x'^2)dx'$$

$$= \frac{bwh}{6}(3x_1 + 2h)$$

2. The *moment* about *CC* of the pressure on the element is $wxydx' \times x'$

and $$M_c = \frac{wb}{h}\int_o^h (x_1x'^2 + x'^3)dx'$$

$$= \frac{wbh^2}{12}(4x_1 + 3h)$$

and $$x_o = \frac{M}{P} = \frac{h}{2}\left(\frac{3h + 4x_1}{2h + 3x_1}\right);$$

x_o is measured from *CC.*

Corollary. If the vertex of the triangle reaches the top of the liquid, $x_1 = 0$, and

$$x_o = \frac{3}{4}h.$$

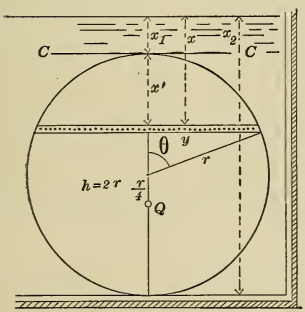


Fig. 115

The center of pressure is shown in Fig. 114, *provided the vertex D is at the surface of the liquid.*

118. Let the surface in the vertical wall of a tank be a circle. Fig. 115. Let x, x_1, x_2, and x' be defined as in the last section. In this case $y = 2\sqrt{x'(2r - x')}$. As intimated in **116**, the total pressure is equal to the area, multiplied by the mean depth.

$$P = (\pi r^2)\cdot\frac{x_2 + x_1}{2}\cdot w, \text{ hence}$$

$$P = \pi w r^2 (x_1 + r)$$

and $$M_c = \int (ydx')\,(wx)x'$$

Changing the variable to θ we have

$$y = 2r\sin\theta$$
$$x' = r\,(1-\cos\theta)$$
$$dx' = r\sin\theta d\theta$$
$$x = x_1 + r\,(1-\cos\theta)$$

and hence

$$M_c = 2wr^3\left[x\int_o^{\pi}\sin^2\theta(1-\cos\theta)\,d\theta + r\int_o^{\pi}\sin^2\theta\,(1-\cos\theta)^2 d\theta\right]$$

$$M_c = 2wr^3\left[x_1\left(\frac{\theta-\sin\theta\cos\theta}{2} - \frac{\sin^3\theta}{3}\right) + r\left(\frac{\theta-\sin\theta\cos\theta}{2} - \frac{2\sin^3\theta}{3} + \frac{2\theta-\sin 2\theta\cos 2\theta}{16}\right)\right]_o^{\pi}$$

$$M_c = 2wr^3\left(\frac{x_1\pi}{2} + \frac{r\pi}{2} + \frac{r\pi}{8}\right) = \pi wr^3\left(x_1 + \frac{5}{4}r\right)$$

and $$x_o = \frac{M_c}{P} = \frac{r}{4}\cdot\frac{4x_1+5r}{x_1+r}$$

Corollary. If the circle touches the surface of the liquid, $x_1 = 0$,

$$x_o = \frac{5}{4}r$$

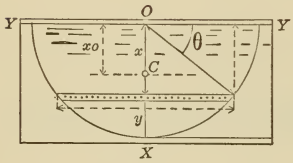


Fig. 116

119. The pressure on a semi-circle. If the circle is half above the surface of the liquid, and only a semi-circle is under pressure, the problem is wholly new.

With the notation shown in Fig. 116

$$x = r\sin\theta$$
$$dx = r\cos\theta\,d\theta$$

and $$y = 2r\cos\theta$$

so that $$P = 2wr^3\int_o^{\frac{\pi}{2}}\cos^2\theta\sin\theta\,d\theta = -\left[\frac{2wr^3\cos^3\theta}{3}\right]_o^{\frac{\pi}{2}} = \frac{2}{3}wr^3$$

and $$M_y = 2wr^4\int_o^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta d\theta = 2wr^4\left[\left(\frac{2\theta-\sin 2\theta\cos 2\theta}{16}\right)\right]_o^{\frac{\pi}{2}}$$

$$M_y = \frac{\pi wr^4}{8} = \frac{\pi r^2}{2}\cdot w\cdot\frac{r^2}{4} = w\frac{\pi r^2}{2}\cdot\frac{r^2}{4}$$

Hence $$x_o = \frac{M}{P} = \frac{3}{16}\pi r$$

120. The student must not fail to see that the *Center of a varying Pressure* is a different point from the *centroid* of the surface. The "Center of Hydrostatic Pressure" is always *below the centroid* of the surface of action, but the distance below depends upon the depth of the liquid. The deeper the liquid, the less the distance between the two points.

The results in all the above cases apply to surfaces in inclined planes which are immersed—if the axis of X is a line of greatest declivity in such a plane. Just as the value of w (the specific weight of the liquid) disappeared from the final formulas for vertical planes, so $w \sin \phi$ will disappear for the inclined plane. Fig. 117.

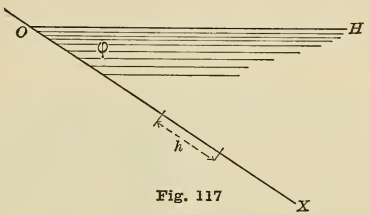


Fig. 117

The practical application of these formulas for hydrostatics is readily seen. If a gate or a valve in the vertical wall of a dam, flume or tank, is to be supported by a single prop, or a pair of shaft bearings, the prop or shaft must be so adjusted as to act normally at the *center of pressure* and *not* at the centroid of the surface. If the gate or valve is to revolve freely on a horizontal axis, *i. e.*, so that the moment pressure will balance on the axis, that axis must pass thru the *center of pressure*.

The problems of this chapter are by no means purely abstract, given simply as exercises. Some of the formulas giving the value of x_o will appear several times in later chapters.

I.

121. The theorems of Pappus. The general expression for the volume generated by the revolution of a plane area about an axis in the plane, not intersecting the area revolved, is (Fig. 118,)

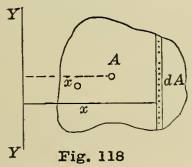


Fig. 118

$$V = 2\pi \int x dA$$

integrated between proper limits; dA is a differential of the given area taken parallel to the axis of revolution. The integral

$$\int x dA = x_o A$$

by the *axioms of moments.*

Hence $$V = 2\pi x_o A$$

In other words the volume of a solid of revolution is equal to the area of the generating surface, multiplied by the path described by the centroid of that surface. This is known as the *Theorem of*

Pappus. It may be used to find a volume when x_o is known, or, conversely, to find x_o when V is known. For example, if we know that the volume of a sphere is $\frac{4}{3}\pi r^3$, we can find the centroid of a semicircle by the equation

$$\frac{4}{3}\pi r^3 = 2\pi x_o \cdot \frac{\pi r^2}{2}$$

$$x_o = \frac{4r}{3\pi}$$

II.

The surface of a solid of revolution is found by multiplying the length of the generating curve by the length of the path of its centroid during the revolution. Fig. 119.

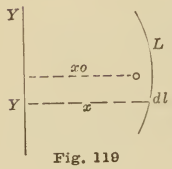


Fig. 119

Since $S = 2\pi \int x dl$, in which S is the surface of a solid of revolution, and l is the length of the generating line, co-planar with the axis, and not intersecting it; and since $x_o l = \int x dl$ we have

$$S = 2\pi x_o l$$

which is the above theorem.

Problems.

1. Find the surface generated by revolving the cycloid about its base.

2. Find the surface and volume of a "torus" generated by revolving a circle about an exterior line in its plane.

3. Find the volume of the solid generated by revolving the spandril of a parabolic half-segment about the axis of the parabola. Fig. 120.

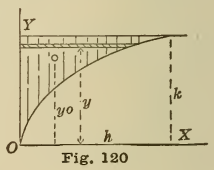


Fig. 120

4. Find the volume of an oblate spheroid whose polar diameter is $2b$, and whose major diameter is $2a$.

CHAPTER VIII.

Centers of Gravity.

122. In Chapter III we studied the relations of parallel forces, both co-planar and in space, without considering the source or nature of the forces. In this chapter we shall deal with parallel forces which are distributed through volumes, such as the earth's attraction

upon all bodies at or near its surface. In Applied Mechanics, we have no occasion to discuss the problems of physical Astronomy. In everything relating to structures and mechanism, the lines of action of the earth's attraction are so nearly parallel that they are assumed to be parallel, and in all cases the arrows representing gravitational forces point vertically downward.

The magnitude of the earth's attraction upon a body depends upon the mass of the body. Since such attraction is known as the *weight** of the body, the weight is proportional to the mass. In a body of uniform density, the masses, and therefore the weights, of equal volumes are equal.

Definition of center of gravity. The earth's attraction upon all of the minutest parts of a body constitutes an infinite number of constituent parallel forces, whose resultant is the weight of the entire body.

It will soon be shown that there is within the enveloping surface of every body a certain point thru which the resultant always passes, no matter how much that body be turned over and about in space. That point is called the Center of Gravity, or the Center of Mass of the body.** This chapter will be devoted to finding the C. G. of typical solids, which closely approximate the forms of real or imaginary bodies.

123. The center of gravity found by experiment. If a body can be suspended by a cord or wire so that the axis of the wire coincides with the resultant line of action of the weight which it balances, we have one line which must contain the C. G. of the body. If, then, the body can be suspended differently, giving a second line of action, the C. G. must be the point where the two lines intersect. This method is frequently used for finding the C. G. of thin plates of irregular shape; for example, a piece of zinc or drawing paper representing the cross-section of a steel rail. In such a case the C. G. of the thin solid is anent the "centroid" of the surface.

124. General formulas. When the bounding surfaces of a solid are given in mathematical language, the co-ordinates of the C. G. are determined as follows:

1. Let the position of the body be given with reference to a set

---

* Under the subject of Deviating Forces in Chap. XVI, it is shown that a component of the earth's attraction is employed in compelling the mass to revolve about the earth's polar axis, and that accordingly the observed *weight* of a body does not measure the entire attraction. But the *weight* is still proportional to the mass.

** In the case of such bodies as are considered in this book, the center of gravity coincides with the center of mass.

of three rectangular axes. Fig. 121. An elementary portion of the body is represented by the rectangular solid

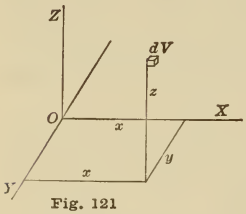


Fig. 121

$$dV = dxdydz.$$

The weight of a unit volume of a homogeneous material is known as its "specific weight." If the specific weight of the material is w, the weight of dV is

$$dW = wdxdydz = wdV$$

If the plane XY is horizontal, the moment of the weight of the element with respect to the axis, OX, is

$$dM_1 = wdV.y = wydxdydz.$$

and

$$M_1 = \int\int\int wydxdydz$$

The successive integrations are to be taken between the limiting surfaces of the solid. If w is constant, it comes before the integral signs. The entire weight of the body is

$$W = w\int\int\int dV = w\int\int\int dxdydz.$$

If the co-ordinates of the C. G. are x_o, y_o z_o, we have by the *axiom of moments*

$$y_o = \frac{M_1}{W}$$

which becomes, if w is constant,

$$y_o = \frac{\int ydV}{V} = \frac{\int\int\int ydxdydz}{\int\int\int dxdydz.}$$

2. Similarly, finding moments about OY we get

$$x_o = \frac{M_2}{W},$$

and

$$x_o = \frac{\int xdV}{V} = \frac{\int\int\int xdxdydz}{V}$$

3. If now the body and the co-ordinate axes be so turned that the plane of XZ is horizontal, and the moments about OX are again considered, we get

$$z_o = \frac{\int zdV}{V} = \frac{\int\int\int zdxdydz}{V}.$$

and the C. G. is found.

If w varies, it does not disappear from the formulas. It is generally assumed to be constant unless the law of its variation is definitely given.

125. Center of gravity by inspection. The center of gravity of a homogeneous body is evidently at its geometrical center, if it has one; like the center of a sphere; the center of a parallelopiped; and the center of a cylinder of revolution, thin, thick or solid, with parallel bases. It is further evident that if a cylinder or prism have parallel bases whose centroids are known, the center of gravity of the solid is at the central point of the right line connecting the centroids of the bases. Moreover, when a body of uniform density has an axis of symmetry, we know that the center of gravity must lie on that axis, and the problem of finding it is rendered simple.

When a body having no geometrical center can be separated into parts each of which has a geometrical center, the center of gravity may be found by the method of moments and "centers of action," as already given in Chapter III. If the number of parts is finite, and the weight and the center of gravity of each part is known, the center of gravity of the whole can readily be found by repeated application of the "axiom of moments."

126. The C. G. of a group. To find the center of gravity of a group of bodies whose weights, and the co-ordinates of whose centers of gravity are respectively known.

N. B. In choosing co-ordinate planes, it is generally best to so take them that one, and if possible two, of the (C. G.)'s lie in the co-ordinate planes.

1. Let a typical body, whose weight is W_1, have a center of gravity at x_1, y_1, z_1. Now assuming that the axis of X is horizontal, and letting x_o be the X-co-ordinate of the center of gravity of the group, we have by the axiom

$$x_o \Sigma W = \Sigma(Wx)$$

Hence
$$x_o = \frac{\Sigma(Wx)}{\Sigma W} = \frac{W_1x_1+W_2x_2+W_3x_3+\text{etc.}}{W_1+W_2+W_3+\text{etc.}}$$

Similarly
$$y_o = \frac{W_1y_1+W_2y_2+W_3y_3+\text{etc.}}{W_1+W_2+W_3+\text{etc.}}$$

$$z_o = \frac{W_1z_1+W_2z_2+W_3z_3+\text{etc.}}{W_1+W_2+W_3+\text{etc.}}$$

The point z_o, y_o, z_o, is, of course, the point thru which the resultant of the forces supporting the entire system against the earth's attraction must act.

2. When integration is necessary, let dV be a differential element of the first order if possible.

127. To find the center of gravity of a solid cone of revolution. 1. Take the axis of the cone as the axis of X with the vertex as the origin. Fig. 122. Take as an element a *lamina** lying between two consecutive planes perpendicular to the axis.

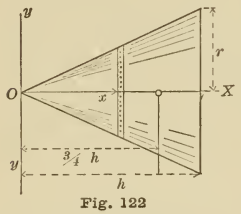


Fig. 122

The center of gravity of the element is in the axis at a distance x from O.

Its volume is $dV = \pi y^2 dx$;

Its weight is $dW = w\pi y^2 dx$;

Its moment about YCY is $dM_y = w\pi y^2 x dx$.

Since $y = \dfrac{rx}{h}$

$$W = \frac{\pi w r^2}{h^2} \int_o^h x 2dx = \frac{\pi w r^2 h}{3}$$

and

$$M_y = \frac{w\pi r^2}{h^2} \int_o^h x^3 dx = \frac{\pi w r^2 h^2}{4}$$

Hence

$$x_o = \frac{M_y}{W} = \frac{3}{4} h.$$

2. The center of gravity of a frustum of a cone of revolution is readily found, by using the limits h_1 and h_2, to be

$$x_o = \frac{3}{4} \cdot \frac{h_2^4 - h_1^4}{h_2^3 - h_1^3}$$

128. An oblique cone. Suppose we have an oblique *cone* or *pyramid*, with a *base of any figure* whose centroid is known, and we wish to find its center of gravity. Let the point C, Fig. 123, be the centroid of the base, and let $VC = x_1$. The line VC will contain the centers of gravity of all elements made by consecutive planes parallel to the base; hence the center of gravity required lies on that line. Let VX and VY be horizontal axes. Let EF be an element parallel and similar to the base, so that we have:

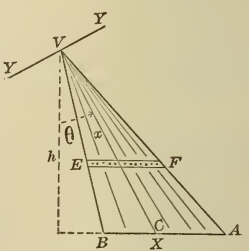


Fig. 123

$$dV = (\text{Area } EF)\ (dx \cos\theta)$$

But

$$(\text{Area } EF) = (\text{Area } AB)\ \frac{x^2}{x_1^2}, \text{ by geometry.}$$

* An infinitely thin plate.

Hence $$W = w\,\frac{\text{Area } AB}{x_1{}^2}\cdot\cos\theta\int_o^{x_1} x^2dx = \frac{h}{3}(\text{Area } AB)w$$

and $$M_y = \frac{hx_1}{4}(\text{Area } AB)w$$

so that $$x_o = \frac{3}{4}\cdot x_1$$

129. Find the center of gravity of a spherical segment. The center of gravity must lie on the axis of X if axes are assumed as shown. Fig. 124. The element is a thin circular plate. As before, we have the general formulas; the segment is $ABDH$.

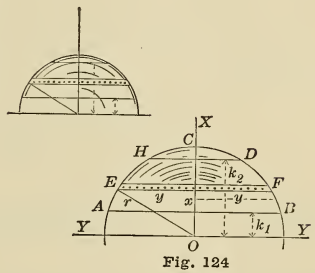


Fig. 124

$$dV = \pi y^2dx$$

$$dW = \pi wy^2dx$$

$$dM_y = \pi wy^2xdx$$

$$W = \pi w\int_{k_1}^{k_2}(r^2-x^2)\,dx = \frac{\pi w}{3}\cdot\left[3r^2(k_2-k_1)-(k_2{}^3-k_1{}^3)\right]$$

$$M_y = \pi w\int_{k_1}^{k_2}(r^2-x^2)\,xdx = \frac{\pi w}{4}\left[(r^2-k_1{}^2)^2-(r^2-k_2{}^2)^2\right]$$

$$x_o = \frac{3}{4}\cdot\frac{(r^2-k_1{}^2)^2-(r^2-k_2{}^2)^2}{3r^2(k_2-k_1)-(k_2{}^3-k_1{}^3)}$$

If the segment have but one base, and $k_2 = r$,

$$x_o = \frac{3}{4}\,\frac{(r^2-k_1{}^2)^2}{2r^3-3r^2k_1+k_1{}^3}$$

If the segment be a hemisphere, $k_1 = o$

$$x_o = \frac{3}{8}r\cdot$$

It is now obvious that the position of the Center of gravity is in no way dependent upon the value of w if it is constant. It will therefore for the present be considered as unity.

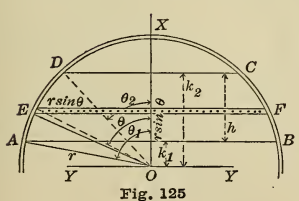


Fig. 125

130. *Problem.* **To find the center of gravity of a segment of a thin spherical shell.** Fig. 125. Let the element be a ring generated by the revolution about OX of the

cross-section $rd\theta dr$, or $trd\theta$. Its center of gravity is on OX at a distance $r\cos\theta$, from YY, and its radius is $r\sin\theta$. Hence

$$dV = (trd\theta)\ (2\pi r\sin\theta)$$

$$W = 2\pi r^2 t \int_{\theta_2}^{\theta_1} \sin\theta\, d\theta = 2\pi r^2 t\ (\cos\theta_2 - \cos\theta_1)$$
$$= (2\pi r)\ (t)\ (k_2 - k_1) = 2\pi rht,$$

where $k_2 - k_1 = h$;
which agrees with a proposition of elementary geometry, which says, that "the area of a segment of spherical surface is equal to the circumference of a great circle $2\pi r$, multiplied by the height of the segment h."

$$M_y = 2\pi r^3 t \int_{\theta_2}^{\theta_1} \sin\theta\cos\theta\, d\theta = \pi r^3 t\ (\sin^2\theta_1 - \sin^2\theta_1)$$
$$= \pi r^3 t\ (\cos^2\theta_2 - \cos^2\theta_1)$$

hence $$x_o = \frac{r}{2}\ (\cos\theta_1 + \cos\theta_2) = \frac{k_2 + k}{2}\ ;$$

that is, the center of gravity of a spherical zone is always in the plane midway between the bounding planes.

Corollary. The center of gravity of a thin hemispherical shell is at $x_o = r/2$, the middle of the central radius.

131. A segment of a thick spherical shell may be made by two parallel planes which cut both spheres; or by two cones which are coaxial and have their vertices at the center of the sphere.

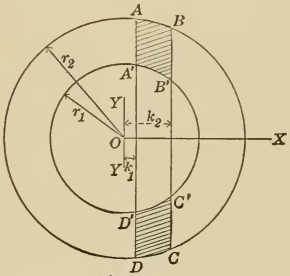


Fig. 126

The first case is seen in Fig. 126. To find the center of gravity of the segment suggested, find W_2 and M_2 for the segment of the larger sphere, as given in **129** above; then W_1 and M_1 for the segment of the smaller spheres. The value of x_o is found by algebra.

$$x_o = \frac{M_2 - M_1}{W_2 - W_1}$$

which involves nothing new.

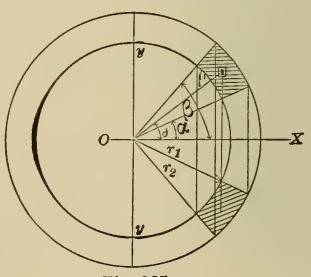


Fig. 127

132. In the second case, Fig. 127, when the segment is cut from the thick shell by two cones, we must integrate an element which has two differential factors. The shaded areas represent the section of the

segment made by an axial plane. The element is a ring whose cross-section is $(\rho d\theta)\ (d\rho)$, and whose circumference is $2\pi\rho\sin\theta$. Hence we have, assuming that $w=1$.

$$dV = 2\pi\rho^2 d\rho \sin\theta\, d\theta$$

$$W = 2\pi \int_{r_1}^{r_2} \int_{\alpha}^{\beta} \rho^2 \sin\theta\, d\rho d\theta$$

The order of integration is immaterial since the limits of one variable are independent of the other variable.

$$W = 2\pi \left(\frac{r_2^3 - r_1^3}{3}\right)(\cos\alpha - \cos\beta)$$

Also

$$M_y = 2\pi \int_{r_1}^{r_2} \int_{\alpha}^{\beta} \rho^3 \sin\theta\cos\theta\, d\rho\, d\theta$$

$$= 2\pi\left(\frac{r_2^4 - r_1^4}{4}\right)\left(\frac{\cos^2\alpha - \cos^2\beta}{2}\right)$$

Hence

$$x_o = \frac{3}{4}\cdot\frac{r_2^4 - r_1^4}{r_2^3 - r_1^3}\cdot\left(\frac{\cos\alpha + \cos\beta}{2}\right)$$

Corollary I.

If $\alpha = 0$ and $r_1 = 0$, we have the center of gravity of a *spherical cone.*

$$x_o = \frac{3}{8} r\ (1 + \cos\beta)$$

Corollary II.

If now $\beta = \frac{\pi}{2}$, the cone becomes a *hemisphere* and $r = \frac{3}{8} r$ as before.

133. *Problem.* To find the center of gravity of the solid cut from the sphere O by the cone V-ABC. Fig. 128. This problem could easily be solved by taking it in two parts: the cone VAB, and the spherical segment ABC (since both volumes and their centers of gravity have been found), and then proceeding by the algebraic method. It may, however, be more simple to proceed directly by integration.

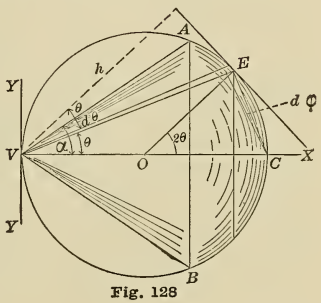


Fig. 128

For an element, pass two co-axial and co-vertical cones whose half-vertical angles are θ and $\theta + d\theta$. Pass also two consecutive axial planes which make with each other an angle $d\phi$. The solid enclosed will be an oblique pyramid whose base is a rectangle $(r\sin 2\theta d\phi)\ (rd2\theta)$, and whose altitude is $2r\cos^2\theta = h$.

Hence $$dW = \frac{1}{3} \cdot 4r^3 \cos^2\theta \sin 2\theta \, d\theta \, d\phi$$

$$W = \frac{8}{3} r^3 \int_o^{\alpha} \int_o^{2\pi} \cos^3\theta \sin\theta \, d\theta d\phi$$

The limits for ϕ are 0 and 2π; for θ they are 0 and α.

$$W = \frac{16\pi r^3}{3} \int_o^{\alpha} \cos^3\theta \sin\theta \, d\theta$$

$$W = \frac{4\pi r^3}{3} (1 - \cos^3\alpha) \qquad (19)$$

The center of gravity of the element is $\frac{3}{4}$ of the slant height from V, or $\frac{3}{4} \cdot 2r\cos^2\theta$ from YY. Hence

$$dM_y = \left(\frac{8}{3} r^3 \cos^3\theta \sin\theta \, d\theta \, d\phi\right) \frac{3}{2} r\cos^2\theta$$

and $$M_y = 8\pi r^4 \int_o^{\alpha} \cos^5\theta \sin\theta \, d\theta$$

$$= \frac{4}{3} \pi r^4 (1 - \cos^6\alpha)$$

Hence $$x_o = \frac{M_y}{W} = \frac{1 - \cos^6\alpha}{1 - \cos^4\alpha} \cdot r$$

Corollary.

If $\alpha = \frac{\pi}{2}$, the solid becomes the sphere, and as $\cos\frac{\pi}{2} = 0$ we have $W = \frac{4}{3}\pi r^3$, and $x_o = r$ which serves as a check on the formula.

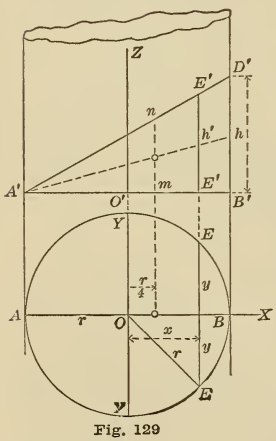


Fig. 129

136. *Problem.* **To find the center of gravity of a wedge cut from a circular cylinder.** The wedge is shown in plan and elevation in Fig. 129. Let the plane of the base be the XY-plane, with the center as the origin, and the plane XZ a plane of symmetry; hence $y_o = 0$. The weight of the wedge is known by inspection to be $W = \frac{1}{2}\pi r^2 h$, if $w = 1$. Taking an element $(2yh'dx)$ its moment about YY is

$$dM_y = 2yxh'dx;$$

but $$h' = \frac{h}{2r}(r+x);\ \text{and}\ y = (r^2 - x^2)^{\frac{1}{2}},$$

hence $$M_y = \frac{h}{r}\int_{-r}^{+r}(r^2-x^2)^{\frac{1}{2}}(r+x)x\,dx = \frac{\pi h r^3}{8}$$

Hence $$x_o = \frac{M_y}{W} = \frac{r}{4}.$$

as was found in Chapter VII for the center of hydrostatic pressure.

1. When we consider the center of a uniformly varying pressure, which is geometrically represented by the wedge-shaped solid ADB (Fig. 126); and the center of gravity of the geometric solid, we find that the center of gravity of the solid and the center of pressure are in the same vertical line.

2. To find Z_o, note the fact that a plane passing thru a tangent to the base at A, and bisecting the altitude h, contains the centers of gravity of all vertical elements, and hence would contain the center of gravity of the entire solid. Hence we have, when $x = \frac{r}{4} = x_o$

$$z_o = \frac{\overline{mn}}{2} = \frac{h}{4r}\left(r + \frac{r}{4}\right) = \frac{5}{16}h$$

This problem is closely analagous to that of finding the center of action of a uniformly varying force. In fact, some writers picture a distributed force whose elements are parallel, as a solid, and find the center of action by finding the center of gravity of the ideal solid.

3. The center of gravity of a truncated cylinder may now be found by the algebraic method.

Problems.

Ex. 1. Find the center of gravity of a semi-circular cylindrical wedge, as shown in Fig. 130. Let $y = r\cos\theta$, $x = r\sin\theta$, $dx = r\cos\theta\,d\theta$.

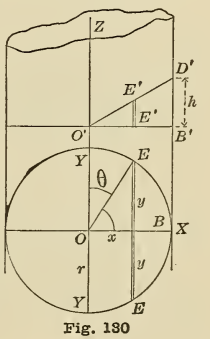


Fig. 130

Ans. $$x_o = \frac{3}{16}\pi r$$
$$z_o = \frac{3\pi}{32}\cdot h.$$
$$y_o = 0$$

Ex. 2. Find the center of gravity of an annular wedge which has for a base one half of a ring cut from a thick cylinder, and whose upper surface is

an inclined plane, which passes thru the diameter of the ring as shown in Fig. 131.

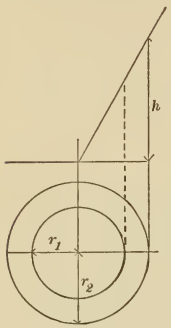


Fig. 131

Ans. $$x_o = \frac{3\pi}{16} \cdot \frac{r_2^4 - r_1^4}{r_2^3 - r_1^3},$$

$$z_o = \frac{3\pi}{32} \cdot \frac{r_2^4 - r_1^4}{r_2^3 - r_1^3} . h$$

137. Composite figures. In the case of a composite figure whose components have geometrical forms, the center of gravity may be found by the algebraic process of addition. It often happens that certain of the components are negative, and that their weight and moment must both be treated as negative quantities. A single example will now be given to illustrate the method when some of the components are negative.

Figure 132 shows a cube whose edges are 3 feet long. All four upper corners are beveled, the bevel being cut away one foot on each of the three edges. A central vertical cylinder is cut out from top to bottom, the diameter being 1.5 feet. The student is to find the volume and center of gravity of the solid.

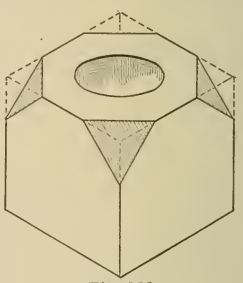


Fig. 132

Suggestions: In finding the volume of one of the pyramids cut off, the student should know that the perpendicular dropped from the vertex of the pyramid upon the equilateral triangular base makes equal angles with the edges, and if θ is the angle which the perpendicular makes with one of the edges, $\cos\theta = \sqrt{\frac{1}{3}}$. See **96**. The student of solid analytic geometry may very readily find the length of a perpendicular upon a plane which has equal intercepts on three rectangular axes.

Problems.

1. Prove that the G. C. of a triangle pyramid is on the right-line from the vertex to the centroid of the base, and one-fourth of the length of that line from the base.

2. Find the C. G. of a fine wire having the form of an arch of the cycloid generated by a rolling circle whose radius is r.

3. Find the center of gravity of a solid generated by revolving a circle about one of its tangents, thru an angle of 180°. Make use of the solution for finding the centroid of a circular arc.

4. Suppose a hemispherical bowl, radius r, of infinitesimal thinness has a plane circular cover, radius r, equally thin, but of twice the density (*i. e.*, its w is twice as great). Find the C. G. of the combination.

CHAPTER IX.

Varying Stress in Beams; Moments of Inertia.

138. Stress and strain. Modulus of elasticity. 1. All materials used in construction are elastic to a greater or less degree. A metallic bar is stretched by tension, shortened by compression. The same is true of wood, stone and cement.

Think of a steel rod of uniform section, A square inches, and length l, subjected to a tension P. Fig. 133. When the bar or rod is well made it is *assumed* that the tension at the cross-section A is uniformly distributed, so that each *square inch* sustains a "*stress*" of $\frac{P}{A}$ lbs. If we write $\frac{P}{A}=p$, we read p as the *unit stress.*

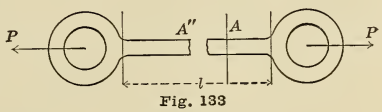


Fig. 133

2. Under this stress the length l becomes $l+\lambda$, if λ is the elongation of the rod, and $\frac{\lambda}{l}$ is the elongation of a unit of length. The elongation λ is called the *strain*, and $\frac{\lambda}{l}$ is called the *unit strain.*

3. Now, so long as p does not exceed a certain limit, called the *elastic limit*, the numerical ratio between p and $\frac{\lambda}{l}$ is constant. This ratio is called "The Modulus of Elasticity" of the steel, and is represented by the letter E, so that

$$E=\frac{unit\ \ stress}{unit\ \ strain}=p\div\frac{l}{l}=\frac{pl}{\lambda};$$

hence

$$\frac{p}{E}=\frac{\lambda}{l}=\epsilon$$

The last form of the definition shows that if there were no limit to the elastic law, **a unit stress of E would double the length of the rod.**

139. How E is found. The value of E is found by actual tests (see **49**), in which p has such values as 6,000 lbs., 10,000 lbs. or 30,000 lbs. A sound bar will sustain such values of p, and when relieved, will recover its exact original length. If, however, a stress

of 80,000 lbs., or 100,000 lbs. were applied, the bar, when relieved, would not fully recover; it would be permanently lengthened, or it would have taken a "permanent set." The greatest value of p without "set" is called the "Elastic limit."* The elastic limit is therefore defined as the stress at which the ratio p/ε ceases to be constant. With greater values of p, the ratio p/ε generally becomes less than E.

The Elastic Limit, or limiting value of p for sound steel varies from 30,000 lbs. to 40,000 lbs. The value of E varies from 28,000,000 lbs. to 30,000,000 lbs. See Appendix.

140. Compressive stress. When a bar or block is subjected to compressive stress, it is shortened, and if the forces are not excessive, the *deformation* λ, will be proportional to p, and we have again a value

$$\frac{\lambda}{l} = \frac{p}{E_c}, \text{ therefore } E_c = \frac{pl}{\lambda}$$

which is the **Modulus of Elasticity for Compression.**

Thus there are two Moduli, E_t and E_c. Under moderate values of p, the two E's for steel are assumed equal, tho their *elastic limits* may vary.

141. Normal stresses. When a cantilever beam is broken by an excessive load, its torn fibers in the top of the beam show that they were pulled in two by a destructive tension; also the crushed and bent fibers in the bottom of the beam show that they suffered from a destructive compression. These features are easily seen if the beam be of wood. A steel beam may not break, but may bend out of shape; still it would show *stretching* on one side and *upsetting* on the other.

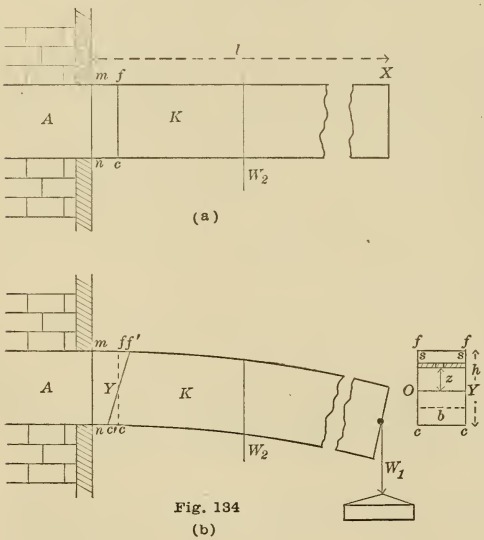


Fig. 134
(b)

Let us suppose now that such a beam, Fig. 134 *a*, is not broken, but is slightly bent, and that a thin layer, or block, *mncf*, is distorted

* A *very slight* "set" may appear upon the first application and removal of a stress below the elastic limit, which set will not be increased by subsequent applications tho repeated many times with even greater stress.

or deformed into $mnc'f'$, Fig. *b*, by the internal stresses. The fibers in its top are lengthened, and those at the bottom are compressed, and the "*strain*" varies on each side of the center (where it must be zero) in such a way that what was a vertical plane face, *fc*, becomes an inclined plane face $f'c'$, and the beam actually bends. This is what happens under a load, and as the *stress* is proportional to the *strain*, it is seen that the *stress varies uniformly* both ways from OY, being tension on the upper half, and compression on the lower, while the horizontal diameter, across the beam, is a "*neutral axis.*" OY is the neutral axis. See cross-section enlarged.

142. The moment of resisting stress. In the cross-section, the rectangle *ffcc* represents the outer face of the thin layer. All the fibers in the area $OffY$, have been subjected to tensile stress; all in the area $OYcc$ have been subjected to compressive stress. This distributed action by the portion of the beam, K, has had a tendency to turn the thin block round *right-handed.* In fact, it *has* turned it a little. It is plain that the portion A must *resist* and *balance* that tendency of K to turn the thin block.

Let us suppose that the intensity of the stress at a unit's distance from the neutral axis OY is a; then the intensity at a point z distant must be az; and the *amount* of *stress* on an element of the surface,

$$dA = bdz, \text{ is } dF = abzdz.$$

And the moment of that stress about the axis OY is

$$dM = abz^2dz$$

so that

$$M = ab\int z^2dz = a\int z^2dA.$$

In the integration the limits of z are $-\frac{h}{2}$, and $+\frac{h}{2}$; so that the integral becomes $\frac{bh^3}{12}$ and

$$M = a\cdot\frac{bh^3}{12}$$

which is known as the **moment of resistance.**

When OY is an axis of symmetry the integration could have been

$$2\int_0^{\frac{h}{2}}.$$

143. The neurtal axis passes thru the centroid of the section. It is readily seen that for every element of the cross-section, dM is

positive, since the only factor in it that can be negative (z), is *squared.* It is otherwise with

$$dF = abz(dz)$$

for z is negative for all points below OY, and it is seen without other proof that, if OY is an axis of symmetry of the surface (the cross-section of the beam), the algebraic sum of the normal stress, tensile and compressive, is zero. In point of fact, F is always zero if the beam be horizontal, and is acted upon by vertical forces alone, no matter what the shape of the cross-section may be, since the pull and the push upon the thin block in the direction of OX must have equal magnitudes. In that case, since

$$F = ab\int zdz = 0$$

we must have

$$\int zabdz = 0$$

which means that the moment of a *uniform* stress on the surface about the axis OY is zero. Hence, the *neutral axis must pass thru the centroid of the surface of action.* This is a very important fact.

144. The bending moment. It should be evident from the above that the tendency of the *external* forces, acting on the beam, to turn it to the right, is, after some bending, fully resisted by the stresses of the fibers *within* the beam. The load tends to turn the thin block about OY one way, while the distributed stress on the rear face, mn, tends to turn it the opposite way, and finally to actually *hold it.* Hence the action of the portion A we have called the holding or "Resisting Moment."

Having found an expression for the resisting moment of the beam in terms of the *internal* stresses of the fibers, let us find the resultant moment of the *external* forces which act upon the beam and so call into action the balancing moment already found. The external forces are W_1, the pull of the chain at the end, and W_2, the weight of the beam whose line of action is vertical thru its center. Their moment about the *same axis, OY*, is

$$M = W_1 l + \frac{l}{2} W_2 \tag{1}$$

This is called the *Bending Moment.*

As the beam, when loaded and slightly bent, is at rest, the two moments must balance, that is

$$a.\ \frac{bh^3}{12} = W_1 l + W_2 \frac{l}{2} \tag{2}$$

from which a, the internal stress at a unit's distance from the neutral axis, can be found as all other quantities are known.

The intensity of stress is greatest at the extreme fibers. In the case of a beam with rectangular cross-section, the greatest stress is

$$p_1 = a.\frac{h}{2}$$

hence

$$a = \frac{2p_1}{h}$$

and the equation of moments becomes

$$p_1\frac{bh^2}{6} = W_2\frac{l}{2} + W_1 l, \tag{3}$$

$$\text{and } W_1 = \frac{bh^2p_1}{6l} - \frac{W_2}{2}$$

from which the maximum allowable load, W_1, can be found if p_1 is the maximum allowable intensity of stress in the fibers of the beam.

Examples.

1. Find the extreme fiber stress in a cantilever steel beam, 24′ long, 2″ wide and 12″ deep if a weight of one ton is hung at its end. (Fig. 135.)

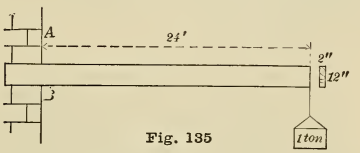


Fig. 135

N. B. Reduce dimensions to inches and the ton to lbs. Steel weighs about 490 lbs. per cubic ft. It is evident from (1) that the Bending Moment, *i. e.*, the moment of the *External* forces, will be greatest when l is greatest. Hence the cross-section we have to consider is at AB, where l is the full length of the cantilever.

Ans. $p_1 = 17{,}880$ lbs. per square inch.

2. If p_1 had been limited to 8,000 lbs., how large could W_1 have been? Ans. $353\frac{1}{3}$ lbs.

3. A beam 14″ by 6″, and 20′ long, supports a uniform load of 180 lbs. per running foot. Fig. 136. The beam is of wood which weighs 38 lbs. per cubic foot. What is the maximum fiber stress in the beam?

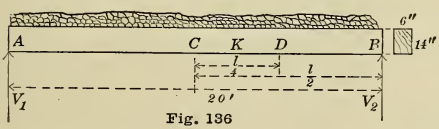


Fig. 136

By symmetry, $V_2 = \frac{1}{2}$ (weight of beam and load). The greatest

stress is evidently at the *center*. The external forces acting on the half K to make it bend about an axis at C are: V_2 = half of the *entire load, acting up at* B; and half of the *entire load*, acting *down* at D;

hence $$p_1 \cdot \frac{bh^2}{6} = \left(\frac{\Sigma W}{2}\right)\left(\frac{l}{4} - \frac{l}{2}\right), \text{ etc.}$$

145. The moment of inertia of plane surfaces. Going back now to **142**

$$M = a\int z^2 (bdz) = a\int z^2 dA$$

it is evident that while a means intensity of stress at a unit's distance from the neutral axis, the integral is only a function of the surface; it is the sum of all the products resulting from multiplying every elementary area by the square of its distance from an axis in its plane. This integral is sometimes called "The Second Moment of the Surface," but the more common name is "The Moment of Inertia" of the surface. It is very generally designated by the capital letter I. If the axis, from which z is measured, passes thru the centroid of the surface, it will be written I_o.

Of course I will vary with the size and shape of the surface, for example, the cross-section of a beam. In the case of a rectangle, we found it to be

$$I_o = \frac{bh^3}{12}$$

We shall soon find values for I_o for triangles, circles and other figures of use in engineering. The general formula which every student should keep in mind is

$$M = aI_o$$

in which M **is the Bending Moment** of external forces with reference to the neutral axis of the cross-section of the beam where the stress is to be examined; and aI **is the Moment of Resistance** of the beam at the same cross section.

146. The radius of gyration. As the name "Moment of Inertia" was borrowed from another department of Mechanics, which is much older in history, so the name "Radius of Gyration" has been taken from the same source. It is here defined simply by the formula

$$k^2 = \frac{I}{A}$$

in which k is the "radius of gyration," and A is the area of the surface under consideration.* I usually can be separated into two factors, one of which measures the area of a cross-section, and the other will be, by definition, the value of k^2. When I is I_o, k becomes k_o.

In the case of the rectangular beam

$$I_o = \frac{bh^3}{12},\ A = bh,\ k_o^2 = \frac{h^2}{12},\ k_o = \frac{h}{6}\sqrt{3}$$

Engineering hand-books usually contain tables giving values of I_o, A, k_o^2 for all beams in ordinary use, and p_1, the maximum allowable stress in the materials used, which greatly facilitates the work of designing or calculating the strength of beams.

147. Take a prismatic beam, whose cross-section is an isosceles triangle, suitably loaded with a uniform load, which, *with the beam itself*, weighs w lbs. per linear foot. Fig. 137. The beam is supported in a horizontal position by piers at the ends. Let us find the maximum fiber stress at a distance x from the end P. The cross-section at A gives the surface of action, which is shown enlarged. Fig. (*b*).

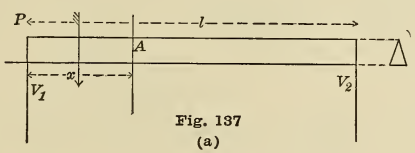
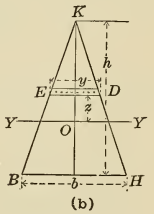


Fig. 137
(a)



(b)

1. Now we know that the greatest stress will be at the top of the beam at the point K, which is distant from the line of no stress, $\frac{2}{3}h$, inasmuch as the neutral axis passes thru the centroid of the triangle. Accordingly

$$p_1 = \frac{2h}{3} \cdot a$$

---

* The name "second moment" would appear to be more logical than the "moment of inertia." The expression zdA is the (so called) moment of an elementary area. When this moment is multiplied by z, we seem to get a moment of a moment, or a "second moment."

However, the term "moment of inertia" is in good use and will be retained as will the mysterious term "radius of gyration." Both will be accounted for later on in this book. All that the students need to know now about either of these terms is, what algebraic functions they represent. $I = \int z^2 dA$ and $k^2 = \frac{I}{A}$.

So we must find a. Now, we know that

$$M = aI_o$$

or

$$a = \frac{M}{I_o}$$

Hence, in order to solve our problem, we must find both M and I_o.

2. M is the moment of the external forces acting on the beam-segment, PA. Those forces consist of the "support" $V_1 = \frac{lw}{2}$ acting *up*; and the weight of PA, which is wx acting *down*, "centered" at the middle point of the segment. Hence, the moment about the *neutral axis at A* is

$$M = \frac{wl}{2}x - wx.\frac{x}{2} = \frac{w}{2}(lx - x^2)$$

We must next find I_o, the Moment of Inertia of the triangular section; see the enlarged figure in Fig. 137.

3. For finding I_o, take an element (ydz).

So that we have

$$I_o = \int z^2 y dz.$$

Since $OK = \frac{2}{3}h$, we get by proportion

$$y = \frac{b}{h} \cdot \left(\frac{2}{3}h - z\right)$$

The limits for integration for z evidently are $-\frac{h}{3}$ and $+\frac{2}{3}h$, so that

$$I_o = \frac{b}{h}\int_{-\frac{h}{3}}^{+\frac{2}{3}h}\left(\frac{2}{3}h - z\right)z^2 dz$$

$$I_o = \frac{bh^3}{36}$$

Now, the extreme fiber is at the point K, distant $\frac{2}{3}h$ from YY, so that

$$p_1 = \frac{2h}{3} \cdot \frac{M}{I_o} = \frac{2h}{3} \cdot \frac{\frac{w}{2}(lx - x^2)}{bh^3} \cdot 36 = \frac{12w}{bh^2}(l - x)x$$

4. It thus appears that the extreme fiber stress varies as does the product of the two segments of the beam: x and $l - x$. It was shown in the calculus that the product of two numbers whose sum was con-

stant was a maximum when the numbers were equal. Hence, the maximum value of p occurs when $x = \frac{l}{2}$ which gives

$$(Max.)p_1 = \frac{3wl^2}{bh^2}$$

148. Engineer's hand book. Of course, we knew, without mathemathics, that the greatest stress (greatest danger of breaking) was at the middle of the beam, and had we turned to the Engineer's hand book we should have found

$$Max.\ \text{Bending Moment} = \frac{wl^2}{8};\ I = \frac{bh^3}{36};$$

$$A = \frac{1}{2}bh;\quad k^2 = \frac{h^2}{18};$$

$$\text{and if we put } c = \frac{2}{3}h, \text{ and } M = \frac{p_1}{c}\cdot Ak^2 = aI_o,$$

we should have had, as found above:

$$\frac{wl^2}{8} = p_1 \cdot \frac{3}{ah} \cdot \frac{bh}{2} \cdot \frac{h^2}{18} \text{ so that } p_1 = \frac{3wl^2}{bh^2}$$

149. Some relative properties of the areas and stresses above and below the neutral axis of a triangle.* The reader has noted the fact that the algebraic sum of the normal stress on a cross-section of a horizontal beam is zero. This is self-evident if the axis YY (the neutral axis) is an axis of symmetry; it does not *look* quite so evident if the section is a triangle as in Figure 137. However, if we find the total stress (tension) above the neutral axis; and then the total compressive stress below the neutral axis, and find them of equal magnitude and with opposite signs, the proof of the doctrine laid down long ago (that the algebraic sum of the horizontal forces must be zero, because the beam is at rest) will be complete.

1. Taking the general formula for the stress, Fig. 137 (*b*)

$$F = a\int yzdz$$

Substituting $$y = \frac{b}{h}\left(\frac{2}{3}h - z\right)$$

* While the discussion which follows may not seem to be very practical, it deals with general principles which are vital, and in an emergency may be important. Engineers have been known to quarrel over matters here discussed.

and integrating for the surface above YY we have

$$F_1 = \frac{ab}{h}\int_o^{\frac{2}{3}h}\left(\frac{2}{3}h - z\right)zdz$$

$$= \frac{ab}{h}\left(\frac{hz^2}{3} - \frac{z_3}{3}\right)\Bigg]_o^{\frac{2}{3}h} = \frac{4abh^2}{81} \qquad (1)$$

2. If now we integrate for the stress below YY, we shall have

$$y = \frac{b}{h}\left(\frac{2}{3}h - z\right) \text{ as before.}$$

(It must be remembered that z is itself negative.)
Accordingly we have

$$F_2 = \frac{ab}{h}\int_{-\frac{h}{3}}^{o}\left(\frac{2}{3}h - z\right)zdz = \frac{ab}{h}\left(\frac{hz^2}{3} - \frac{z^3}{3}\right)\Bigg]_{-\frac{h}{3}}^{o}$$

$$= -\frac{4abh^2}{81}$$

and $F_1 + F_2 = 0$, as was expected.

The proposition is universally true if the neutral axis passes thru the centroid of the surface.

3. But the **Moments** of the stresses above and below **are not equal** for the triangle, as is readily shown.

Taking the formula

$$M = a\int yz^2dz$$

and integrating for the moment of the stress above YY, we have

$$M_1 = \frac{ab}{h}\int_o^{\frac{2}{3}h}\left(\frac{2}{3}h - z\right)z^2dz = \frac{4}{243}abh^3$$

4. Integrating for the moment of the stress *below* the neutral axis we have

$$M_2 = a\frac{b}{h}\int_{-\frac{h}{3}}^{o}\left(\frac{2}{3}h - z\right)z^2dz = \frac{ab}{h}\left(\frac{2h}{3}\cdot\frac{z^3}{3} - \frac{z^4}{4}\right)\Bigg]_{-\frac{h}{3}}^{o}$$

$$= \frac{11}{972}abh^3$$

so that $\frac{M_1}{M_2} = \frac{16}{11}$, M_1 being the larger.

Since a is the same for the both areas, $\frac{I_1}{I_2} = \frac{16}{11}$ (23)

The sum $M_1+M_2=M=a\frac{bh^3}{36}$

and $$I_1+I_2=I_o=\frac{bh^3}{36}$$

as already found.

5. The "Center" of the uniformly varying stress *above* the neutral axis is

$$z_1=\frac{M_1}{F_1}=\frac{\frac{4}{243}abh^3}{\frac{4}{81}abh_2}=\frac{h}{3}$$

which is just half-way from YY to K.

6. The "Center" of the uniformly varying stress *below* the neutral axis is

$$z_2=\frac{M_2}{F_2}=\frac{\frac{11}{972}abh^3}{-\frac{4}{81}abh^2}=-\frac{11}{48}h$$

so that, numerically,

$$\overline{C_1C_2}=\frac{h}{3}+\frac{11h}{48}=\frac{9}{16}h=L$$

It must now be evident that the internal normal stress of a beam forms a couple at every section whose force is F_1, whose lever arm is L, and whose moment is

$$F_1L=\frac{4ahb^2}{81}\times\frac{9}{16}h=\frac{abh^3}{36}=M=aI_o$$

as already found.

150. Results may be summarized.

1. A_1 is to A_2 as 4 is to 5.
2. $F_1=F_2$.
3. M_1 is to M_2 as 16 to 11.
4. I_1 is to I_2 as 16 to 11.
5. $OC_1=\frac{h}{3}$ center of stress on A_1.
6. $OC_2=\frac{11}{48}h$, center of stress on A_2.
7. $OC_1+OC_2=L=\frac{9}{16}h$.
8. $LF=M_1+M_2=M=a\frac{bh^3}{36}$

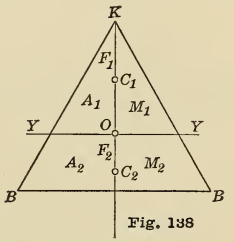


Fig. 138

A similar set of results and comparisons can, of course, be made for every cross-section of a beam whenever the areas on the two sides of the neutral axis are unequal, which is often the case, because the material used is not as strong and elastic in tension as it is in compression, as, for instance, when the material on one side of the neutral axis is concrete and on the other side of the neutral axis is steel, or a combination of steel and concrete. Such cross-sections will be considered in later sections.

151. This chapter does not aim to cover the extensive subject of beams. Some simple forms have been introduced to illustrate the subject of uniformly varying stress, and to develop the function, "Moment of Inertia," and show how intimately connected it is with the study of stress and the strength of beams.

1. Before solving and proposing a variety of problems for finding values of I, it may be well to state again some assumptions we are making, and to put down in words and in the language of mathematics, the general equations we are to apply.

2. All beams and girders are assumed to be of homogeneous material, whatever may be the cross-section, and to have uniformly varying normal stress at every cross-section, when they are slightly bent under moderate loads. In all cases this important formula holds:

The Bending Moment of external forces acting to bend the beam at any section is numerically equal to the Moment of Resistance of the internal stresses acting at that section.

If p is the stress in the extreme fiber, c the distance of the extreme fiber from the neutral axis, and M the Bending Moment, the above dictum is more accurately expressed as follows:

$$M = aI_o = \frac{p}{c} I_o = \frac{pA}{c} \cdot k_o^2$$

3. The general formula for the Moment of Inertia when YY is the axis of reference is

$$I = \int\int z^2 dz dy.$$

When, however, the element of the surface can be a differential strip parallel to the Y-axis, the formula is greatly simplified by assuming the element ydz, so that we have

$$I = \int z^2 y dz.$$

152. Moments of Inertia, fundamental examples.

1. Find I_o for a circle whose radius is r. Fig. 139. The element of the surface is ydz

$$I_o = 2\int_o^r z^2 y dz$$

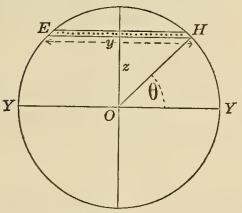


Fig. 139

Introducing θ, we have

$$y = 2r\cos\theta. \quad z = r\sin\theta. \quad dz = r\cos\theta d\theta$$

$$\text{and } I_o = 2r^4 \int_o^{\frac{\pi}{2}} \sin^2\cos^2\theta d\theta$$

$$= \frac{r^2}{2}\int_o^{\frac{\pi}{2}} \sin^2 2\theta d(2\theta)$$

$$I_o = \frac{\pi r^4}{4} = A\frac{r^2}{4} = Ak_o^2, \text{ so that } k_o^2 = \frac{r^2}{4} \tag{25}$$

For a *semi-circle* about its base

$$I = \frac{\pi r^4}{8}$$

2. The I_o of a Circular Ring, radii r_1 and r_2.

$$I_o = \frac{\pi r_2^4}{4} - \frac{\pi r_1^4}{4} = \frac{\pi}{4}(r_2^4 - r_1^4) = A.\frac{r_2^2 + r_1^2}{4}$$

3. Find I_o for an ellipse.

Let the equation of the ellipse be $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and let the axis of reference be YY.

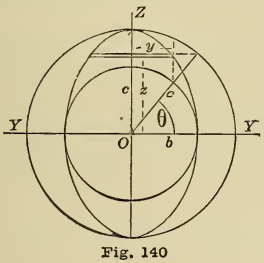


Fig. 140

From Fig. 140, it is evident that

$$I_o = 2\int_o^c z^2 y dz$$

Introducing θ, we have

$$y = b\cos\theta. \quad z = c\sin\theta. \quad dz = c\cos\theta d\theta$$

so that

$$I_o = 4c^3 b \int_o^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta$$

$$I_o = \frac{\pi c^3 b}{4} = A.\frac{c^2}{4} = Ak_o^2 \therefore k_o = \frac{c}{2} \tag{26}$$

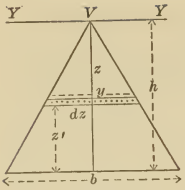


Fig. 141

4. The Moment of Inertia of a triangle about an axis *thru its vertex* parallel to the base. Fig. 141.

The element is ydz, in which

$$Y = \frac{b}{h} \cdot z$$

hence
$$I_v = \frac{b}{h}\int_o^h z^3 dz = \frac{bh^3}{4} = A.\frac{h^2}{2} = Ak^2. \quad \therefore k = \frac{h}{\sqrt{2}} \tag{28}$$

5. Had we used *the base as the axis* of reference, we should have had (Fig. 141).

$$I_b = \frac{b}{h}\int_o^h (h-z')z'^2 dz = \frac{b}{h}\left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{bh^3}{12} \tag{29}$$

$$= A.\frac{h^2}{6} = Ak^2$$

7. The relations of the three values of I for a triangle are thus shown in Fig. 142.

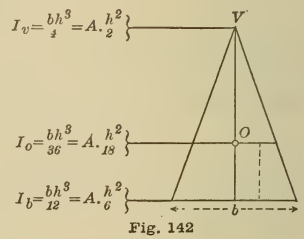


Fig. 142

153. An exterior axis in the plane. The Moment of Inertia of a surface with reference to an axis not passing thru the centroid may be found directly by integration, as has just been shown, but it is often more convenient to make I depend upon I_o for a parallel axis. Thus:

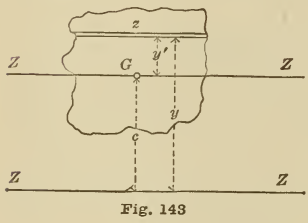


Fig. 143

To find I for the axis ZZ, Fig. 143:

(so that $dy = dy'$). $$I_z = \int y^2 z dy, \text{ but } y = c + y'$$

If Z_oZ_o is parallel to ZZ, and the distance from ZZ to the centroid is c, Hence

$$I_z = \int (c^2 + 2cy' + y'^2) z dy'$$
$$= c^2\int z dy' + 2c\int y'(z dy') + \int y'^2 z dy'$$
$$= c^2 A + I_o$$

Since the first integral is plainly the area of the given surface; the second integral is the moment of a surface about its own centroid axis, which must be zero; and the last is merely I_o. Hence the rule:

$$I_z = I_o + Ac^2 \tag{1}$$

The Moment of Inertia of a surface with reference to any axis in its plane is equal to I_o for a parallel axis thru the centroid of the surface, *plus* the product of the given area multiplied by the square of the distance between the two axes.

1. Thus the I_v for the triangle in **153** could have been found from the knowledge of

$$I_o = \frac{bh^3}{36}$$

for
$$I_v = \frac{bh^3}{36} + \frac{bh}{2} \cdot \left(\frac{2}{3}h\right)^2 = bh^3\left(\frac{1}{36} + \frac{8}{36}\right) = \frac{bh^3}{4}$$

2. **Conversely,** knowing I_z, I_o for a parallel axis can be found from the formula

$$I_o = I_z - Ac^2 \qquad (2)$$

3. For a rectangle about its base we have, since $I_o = \frac{bh^3}{12}$

$$I_b = \frac{bh^3}{12} + \frac{h^2}{4} \cdot bh = \frac{bh^3}{3}$$

4. For a *semicircle* about Z_oZ_o Fig. 144,

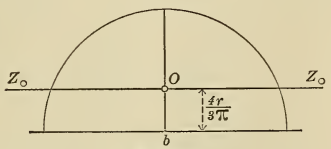


Fig. 144

$$I_o = I_b - A\left(\frac{4r}{3\pi}\right)^2.$$

Hence
$$I_o = \frac{\pi r^4}{8} - \frac{\pi r^2}{2} \cdot \frac{16r^2}{9\pi^2} = \frac{r^4}{72\pi}(9\pi^2 - 64) = A.\left(\frac{9\pi^2 - 64}{36\pi}\right)r^2 \qquad (3)$$

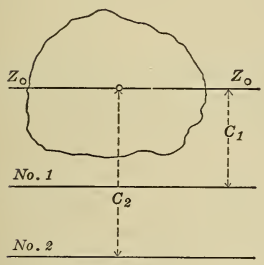


Fig. 145

5. Finally, knowing I and c for any axis, we can find I for any *known parallel co-planar axis.* Suppose we have I_1 and c for No. 1; and we wish to find I_2 for No. 2. Fig. 145.

$$I_o = I_1 - A_{c1}{}^2$$
$$I_2 = I_o + A_{c2}{}^2 = I_o + A(c_2{}^2 - c_1{}^2)$$

154. The Moment of Inertia of a composite surface. It is first necessary to find the centroid of the surface, then I_o is found by use of the formula:

$$I_o \text{ for } Z_oZ_o = \Sigma I_o + \Sigma(Ac^2)$$

The addition in practice is algebraic, since both I_o and A may be negative, as will now be illustrated. See Fig. 146.

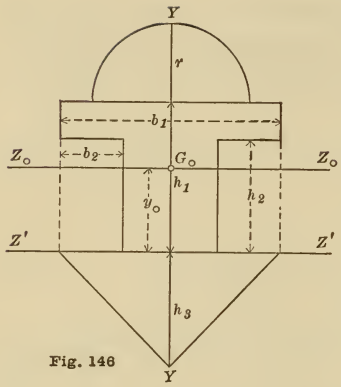


Fig. 146

Example. The line YY is an axis of symmetry, and must contain the centroid G_o. Take any convenient line as a preliminary Z-axis as $Z'Z'$.

The total area is

$$\Sigma A = \frac{\pi r^2}{2} + b_1h_1 - 2b_2h_2 + \frac{1}{2}b_1h_3$$

The total moment with reference to $Z'Z'$ is

$$\Sigma(Ay_o) = \frac{\pi r^2}{2}\left(h_1 + \frac{4r}{3\pi}\right) + \frac{b_1h_1^2}{2} - b_2h_2^2 - \frac{b_1h_3^2}{6}$$

In the case of the two absent rectangles, both areas and moments are negative. In the case of the triangle the area is positive but the moment is negative. Hence

$$y_o = \frac{\Sigma(AY_o)}{\Sigma A} = \frac{\pi r^2\left(h_1 + \frac{4r}{3\pi}\right) + b_1h_1^2 - 2b_2h_2^2 - \frac{b_1h_3^2}{3}}{\pi r^2 + 2b_1h_1 - 4b_2h_2 + b_1h_3}$$

Having found and located G_o, find the I_o for the areas separately, with reference to their own centroid axes. Then find ΣAc^2 and add results.

Numerical example—

Let $r = 2$, $b_1 = 6$, $h_1 = 4$, $h_2 = 3$, $h_3 = 3$.

Find the position of G_o and the value of I_o.

155. Built up beams, struts and columns or posts are very frequently made of separate steel bars or plates securely riveted together so that the cross-section consists of parts, and the moment of inertia must be found as in the last example.

When a post or column is used to carry a vertical load, its *stiffness*, whereby it resists lateral bending, is an important element in its strength; and its moment of inertia is needed with reference to more than one axis thru the centroid of a section. For example, in the case of an ellipse (Fig. 140) we have

$$I_y = \int z^2 dA, \text{ and } I_z \int y^2 dA$$

N. B.—When two Moments of Inertia are found for the same surface, about different rectangular axes, with O at the centroid,

the *larger* will be called I or I_1, and the *smaller* may be called J or I_2. Thus for the ellipse whose major axis is $2a$, and whose minor axis is $2b$,

$$I_1 = \frac{\pi a^3 b}{4} \text{ and } J = \frac{\pi a b^3}{4}.$$

156. Commercial shapes of rolled steel. In determining shapes of rolled structural steel, exact dimensions are given including rounded corners and fillets, as is shown in Fig. 147; but in computing Moments of Inertia for strength and weight they are often ignored or their fillets are assumed to balance the rounded corners, as in Figs. 148-9, which are taken with notation and formulas from the "Carnegie Pocket Book."

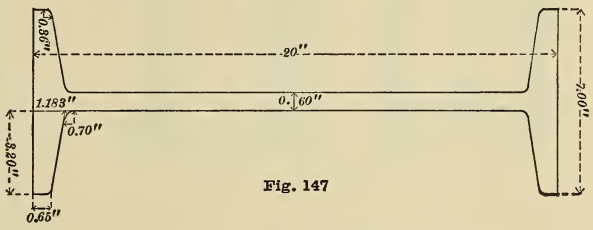


Fig. 147

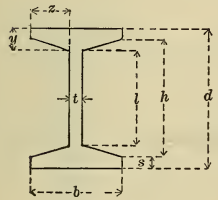


Fig. 148

Let the student check the values of A, I and J.

I = Mom. In. for a neutral axis *parallel* to a *flange*.

J = Mom. In. for a neutral axis *parallel* to a *web*.

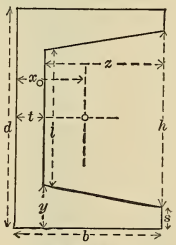


Fig. 149

An I-Beam. Fig. 148.

$$\text{Area} = A = dt + (s+y)2z$$

$$I = \frac{1}{12}\left[bd^3 - \frac{3}{2}(h^4 - l^4) \right].$$

$$J = \frac{1}{12}\left[b^3(d-h) + lt^3 + \frac{1}{24}(b^4 - t^4) \right].$$

A Channel Section. Fig. 149.

$$\text{Area} = dt + (s+y)z$$

$$x_o = \frac{1}{A}\left[b^2 s + \frac{1}{2}ht^2 + \frac{1}{18}(b-t)^2(b+2t) \right]$$

$$I = \frac{1}{12}\left[bd^3 - \frac{3}{4}(h^4 - l^4) \right]$$

$$J = \frac{1}{3}\left[2sb^3 + lt^3 + \frac{1}{12}(b^4 - t^4) \right] - Ax_o^2$$

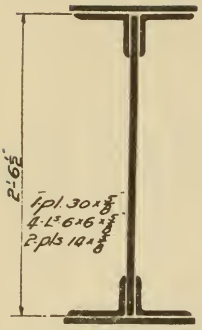


Fig. 150

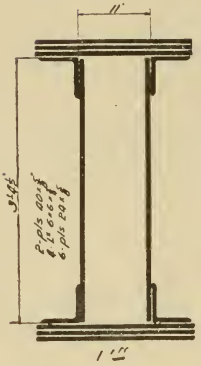


Fig. 152

Fig. 151 consists of: a Web-Plate 60″x⅞″; each Flange has 2 Angles 6″x6″x⅝″, and 3 Plates 14″x⅝″.

Fig. 153 is a cross-section of a Column of the Chicago & N. W. R'y Office Building, Chicago; 6 Plates 16″x⅝″; 2 Plates 16″x$1\frac{1}{16}$″; 2 Plates $12\frac{1}{2}$″x¾″; 4 Angles 6″x6″x¾″.

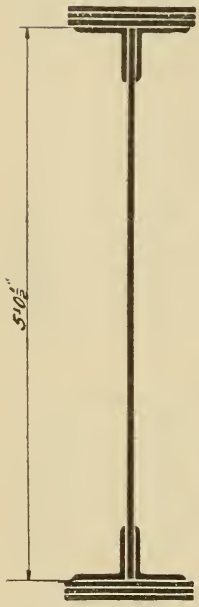


Fig. 151

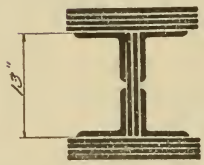


Fig. 153

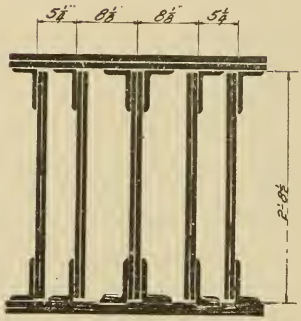


Fig. 154

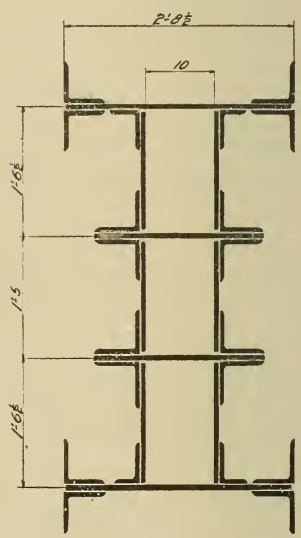


Fig. 155

Fig. 154 is a cross-section of Column 280, Waldorf-Astoria Hotel, New York. It consists of: 10 Plates 32½″x⅝″; 4 Plates 36″x¾″; 4 Angles 6″x4″x$1\frac{1}{16}$″; 8 Angles 6″x3½″x⅝″.

Fig. 155 is a cross-section of a Column in Chicago Steel Co. Building, Chicago. It consists of: 2 Plates 32″x½″; 2 Plates 23″x⅜″; 4 Plates 18″x$\frac{7}{16}$″; 2 Plates 16″x$\frac{7}{16}$″; 4 Angles 6″x6″x⅝″; 12 Angles 6″x6″x$\frac{7}{16}$″; 4 Angles 6″x4″x½″.

Fig. 156, is a cross-section of the lower chord in the Quebec Bridge **which failed during the erection.**

It consisted of materials as follows:

Outer Ribs: 2 Plates $54''x13\frac{3}{16}''$
Each 1 Plate $54''x\frac{7}{8}''$
1 Plate $37\frac{3}{4}''x15\frac{5}{16}''$
2 Angles $8''x6''x15\frac{5}{16}''$

Inner Ribs: 3 Plates $54''x13\frac{3}{16}''$
Each 2 Plates $46''x15\frac{5}{16}''$
2 Angles $8''x3\frac{1}{2}''x15\frac{5}{16}''$

Lattices across the top and bottom of all four ribs.

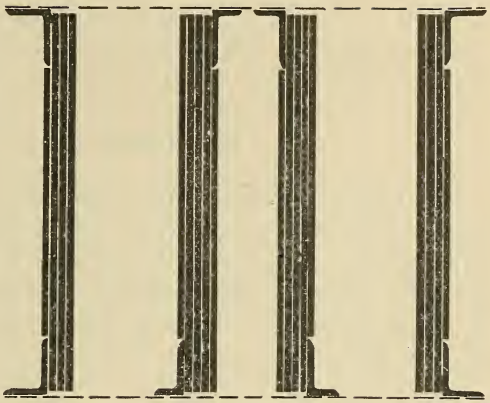


Fig. 156

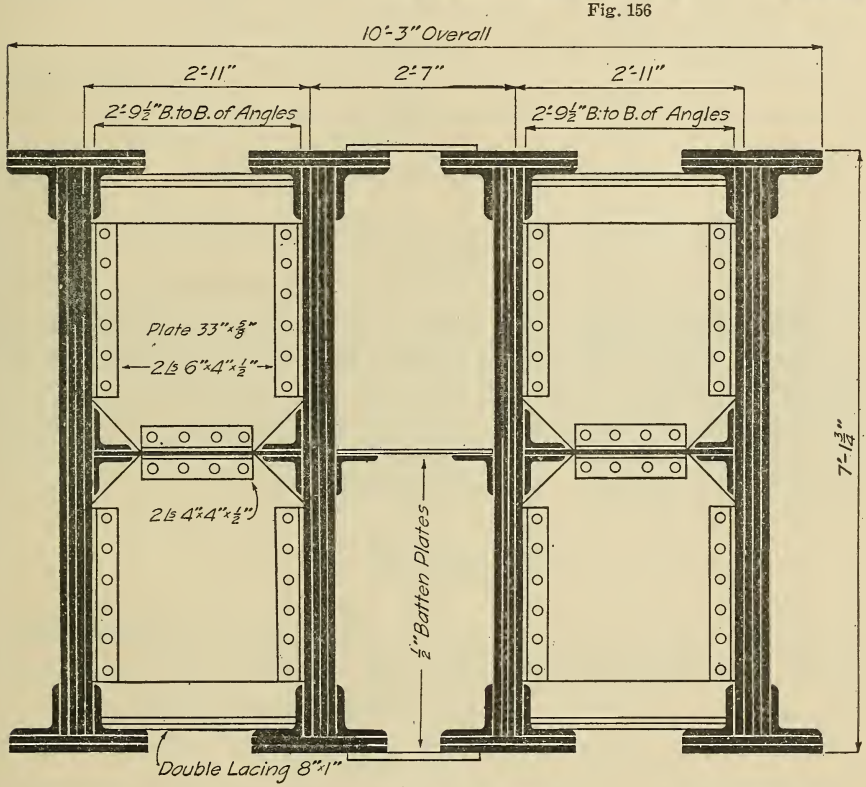


Fig. 157

Fig. 157 is a cross-section of the lower chord of the **new Quebec Bridge,** as designed for the position of the one that failed. The exquisitely drawn cut, is almost self-explanatory. However, the section requires:—

8 Plates $81.5''x\frac{3}{4}''$
8 Plates $81.5''x1\frac{1}{8}''$
16 Plates $20''x1''$ for flanges
16 Angles $8''x8''x1''$ for ribs
10 Angles $6''x6''x\frac{5}{8}''$ for diaphragm plates
2 Plates $33''x11\frac{1}{16}''$ for diaphragms
4 Plates $33''x\frac{5}{8}''$ vertical between ribs
16 Angles $6''x4''x\frac{1}{2}''$ for vertical plates
8 Angles $4''x4''x\frac{1}{2}''$ for diaphragms
4 Lattice Lacings $8''x1''$
3 Plates $\frac{1}{2}''$ thick horizontal between central ribs

157. A variety of examples showing cross-sections of beams and struts in actual use is given above. The moment of inertia of each one may be found from the dimensions given.

Remarks upon Figs. 156 and 157.

The chord member which failed, $A9L$, had lattice bracing along the edges of the ribs as shown by the broken lines. The purpose of this bracing was to prevent the buckling of the ribs sideways. Prof. Geo. F. Swain, of Havard, who made a careful study of the design, is of the opinion that failure was primarily due to the inadequacy of that bracing.

CHAPTER X.

Moments of Inertia—(Continued).

The Relation Between Strength and Moment of Inertia.

158. The strength of a beam is measured by the load it can carry with safety. The test of strength may come from a load at the end of a cantilever, or at the center of a beam supported on piers. In either case the bending moment is proportional to the load; hence M, the bending moment, or its equal, the Moment of Resistance, is a measure of its strength. It was found that

$$M = \frac{p_1}{c} I_o$$

in which c is the distance from the neutral axis to the extreme fiber, and p_1 is the greatest intensity allowed. It follows that the strength of a beam is *directly proportional to* I, and *inversely proportional* to c, p_1 being the same.

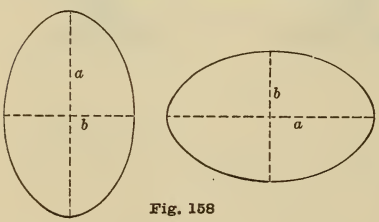


Fig. 158

1. To illustrate, suppose a beam has an elliptical cross-section and that it is loaded first with its major axis vertical, and second with that axis horizontal. Fig. 158.

We have $$M_1 = \frac{p_1}{a} \cdot \frac{\pi a^3 b}{4} = p_1(\pi ab) \cdot \frac{a}{4}$$

$$M_2 = \frac{p_1}{b} \cdot \frac{\pi a_1 b^3}{4} = p_1(\pi ab)\frac{b}{4}$$

Hence $$\frac{M_1}{M_2} = \frac{a}{b}$$

or the strengths are to each other as the semi-axes.

2. Again take a rectangular beam. Fig. 159. When it is "edgeways"

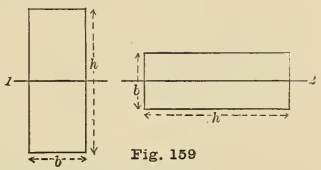


Fig. 159

$$M_1 = \frac{2p_1}{h} \cdot \frac{bh^3}{12} = p_1(hb)\frac{h}{6}$$

When "flatways"

$$M_2 = \frac{2p_1}{b} \cdot \frac{hb^3}{12} = p_1(hb)\frac{b}{6}$$

so that

$$\frac{M_1}{M_2} = \frac{h}{b}$$

This means that a floor joist 2″x12″ will safely support six times as much when on edge as when flat. This is abundantly confirmed by personal experience. Proper *lateral* support is assumed.

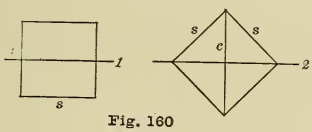


Fig. 160

3. A third case is hardly a matter of experience. Suppose a square beam is placed with sides vertical. Fig. 160. Its strength is

$$M_1 = \frac{2p_1}{s} \cdot \frac{s^4}{12} = p_1(s^2)\frac{s}{6}.$$

If its diagonal is vertical,

$$M_2 = \frac{p_1}{c} \cdot 2 \cdot \frac{2cc^3}{12} = p_1(2c^2)\frac{c}{6}$$

$$\frac{M_1}{M_2} = \frac{s}{c},$$

or the strength of No. 1 is 1.4 times that of No. 2.

The unmathematical reason is plain: In No. 1 the whole extreme *layers* of fiber are utilized for the stress p_1; in No. 2, only a single fiber

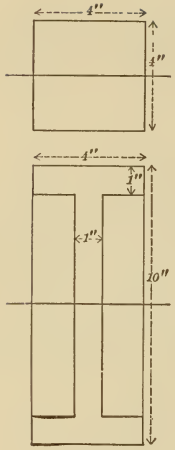


Fig. 161

on the top and the bottom has the stress p_1 while the increased material near the horizontal diagonal adds very little to the strength.

156. The moral derived from the above illustrations is, to put as much of the material as far from the neutral axis as possible, leaving near the neutral axis only enough to preserve the unity and stability of the beam. This will be illustrated by two ideal examples.

1. Compare the strength of a steel bar 4″ square with an I-beam made of the *same amount of steel*, the flanges being 1″x4″, and the web 1″x8″. Fig. 161.

$$M_1 = p_1 \frac{bh^2}{6} = p_1 10\frac{2}{3} \text{ for No. 1.}$$

$$M_2 = p_1 \cdot \frac{1}{5} \cdot \left(\frac{4(10)^3}{12} - \frac{3.(8)^3}{12} \right) = p_1 41\frac{1}{15} \text{ for No. 2.}$$

Hence $$\frac{M_2}{M_1} = \frac{41\frac{1}{15}}{10\frac{2}{3}} = 4 \text{ nearly.}$$

or, **the I-beam with the** *same weight* **of material is almost** *four times* **as strong.**

2. A certain load, W, is to be carried by a cantilever beam of length l. Compare the *weight* of a beam in the shape of a solid cylindrical rod, with the *weight* of a cylindrical tube of the same length, with an exterior diameter n times as great, and *equally strong.*

The expression for the strength of a solid cylinder is

$$M_c = p_1(\pi r^2)\frac{r}{4} = p_1 A_1 \frac{r}{4}$$

If the exterior radius of the tube is nr, and the inner radius x, we have,

$$M_t = \frac{p_1}{nr} \cdot (I_{nr} - I_x) = \frac{p_1}{nr} \cdot \frac{\pi}{4}(n^4r^4 - x^4)$$

Equating the two expressions for M, since they are equally strong, and solving for x^2, we have

$$x^2 = r^2\sqrt{n^4 - n}$$

Hence, the area of the ring, which is the cross-section of the tube, is

$$A_2 = \pi(n^2r^2 - r^2\sqrt{n^4 - n})$$

Since the weights of two beams of the same length are to each other as the areas of their cross-sections, we have.

$$\left.\begin{aligned}\frac{\text{Weight of solid cylinder}}{\text{Weight of tube}} = \frac{A_1}{A_2} &= \frac{\pi r^2}{\pi r^2(n^2-\sqrt{n^4-n})} \\ &= \frac{2n}{1} \text{ nearly}\end{aligned}\right\}$$

(since $\sqrt{n^4-n} = n^2 - \frac{1}{2n} - \frac{1}{8n^4}$ etc.), which means that:—

If the tube's exterior diameter is $\left\{\begin{matrix}\textbf{twice}\\ \textbf{thrice}\end{matrix}\right\}$ **as great as that of the solid cylinder, and** *equally strong*, **it will weigh only one** $\left\{\begin{matrix}\textbf{fourth}\\ \textbf{sixth}\end{matrix}\right\}$ **as much; and so on:** under *ideal* conditions, which would be quite *unreal* if n be made too large.

The saving of weight while preserving strength, thru the use of tubes, is the secret of light-framing and light-rolling stock. For a discussion of the *stiffness* of hollow shafting; see Chapter XIX.

157. The axis for the maximum value of I_o. The student is now prepared to see that while an unsymmetrical surface may have an indefinite number of values of I for different axes across it, there always must be one greatest I_o, and one least. In engineering practice both the greatest and the least are very important. It is now necessary to show how they are found and valued.

Let O, Fig. 162, be the centroid of a plane surface, and OZ and OY a set of rectangular axes so taken that I_z and I_y can be, and have been, found. Let the larger be I_z. Let OZ_1 and OY_1 be a second set of rectangular axes meeting at O, and determined by the angle θ. Let P mark the position of an element of the surface, d^2A, whose co-ordinates are (z,y) and (z_1, y_1) for the respective axes.

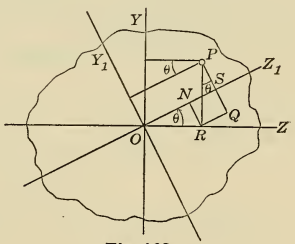


Fig. 162

The quantities $I_z = \int\int y^2 d^2A$ and $I_y = \int\int z^2 d^2A$ are supposed to have been found. The surface element d^2A may be thought of as $dydz$, or $dy'dz'$.

From the figure,

$$\left.\begin{aligned}y_1 &= PQ - RN = y\cos\theta - z\sin\theta \\ z_1 &= RQ + ON = y\sin\theta + z\cos\theta\end{aligned}\right\}$$

Hence $I_{z1} = \int\int y_1^2 d^2A = \int\int (y^2\cos^2\theta - 2yz\,\theta\sin\cos\theta + z^2\sin^2\theta) d^2A$

$$I_{z1} = \cos^2\theta I_z - 2\sin\theta\cos\theta \int\int yz d^2A + \sin^2\theta I_y.$$

Similarly $I_{y1} = \sin^2\theta I_z + 2\sin\theta\cos\theta \int\int yz d^2A + \cos^2\theta I_y.$

The integral $\int\int yz d^2A$ is sometimes called (for the sake of a name) the *Product of Inertia.* More conveniently it will be called simply K

$$K = \int\int zy d^2A.$$

When necessary its value will be found by integration. If either OZ or Oy is an axis of symmetry, K is zero, since the differential terms in the integral are in pairs which balance.

Therefore, we have

$$I_{z1} = \cos^2\theta I_z - 2\sin\theta\cos\theta K + \sin^2\theta I_y \tag{1}$$

$$I_{y1} = \sin^2\theta I_z + 2\sin\theta\cos\theta K + \cos^2\theta I_y \tag{2}$$

Adding, we have

$$I_{z1} + I_{y1} = I_z + I_y \tag{3}$$

Knowing I_z and I_y, and having calculated I_{z1}, I_{y1} can be found from the last equation.

1. Take for example the simple case of an ellipse and suppose we want I_{z1}.

We know that $$I_z = \frac{\pi ba^3}{4}, \text{ and } I_y = \frac{\pi ab^3}{4}$$

and by inspection we see that $K = 0$.

Let $$\theta = \frac{\pi}{4}$$

Then $$I_{z1} = \frac{I_z + I_y}{2} = I_{y1} = \pi ab \cdot \frac{a^2 + b^2}{8}$$

Interesting values follow if $\theta = \frac{\pi}{6}$ and if $\theta = \frac{\pi}{3}$.

Going back to Eq. 1, we see that the value of I_{z1} varies as θ varies, and will have either a maximum or a minimum value when $\frac{dI_{z1}}{d\theta}$ is zero. Hence, differentiating (1) and remembering that I_z, I_y and K are constants, we have

$$\frac{dI_{z1}}{d\theta} = -2\sin\theta\cos\theta I_z + 2(\sin^2\theta - \cos^2\theta)K + 2\sin\theta\cos\theta I_y$$

or $$\frac{dI_{z1}}{d\theta} = -2\sin\theta\cos\theta(I_z - I_y) + 2(\sin^2\theta - \cos^2\theta)K$$

Placing the second member equal to zero so as to find the values of θ for the greatest and least values of I_z, we have

$$(I_z - I_y)\sin 2\theta = -2\cos 2\theta . K$$

$$\tan 2\theta = \frac{-2K}{I_z - I_y} \qquad (4)$$

If $K = 0$: $2\theta = 0$ and π;

or $\theta = 0$ and $\frac{\pi}{2}$, hence in the case of the ellipse, the major and minor axes of the ellipse are the axes for minimum and maximum values of I.

158. When K is not zero. The angle 2θ always has two values differing by π; hence θ has two values differing by $\frac{\pi}{2}$, one acute, the other obtuse, so that the axes for greatest and least values of I are at right angles. In all cases, since $I_z > I_y$, the axis for maximum I, makes an angle numerically less than 45° with the original Z-axis. The axes for $I(\text{max.}) = I$; and I (min.) $= J$, are called the *Principal Axes of the Surface.*

Since $K = 0$ for an axis of symmetry, it follows that that axis and its perpendicular thru the centroid *are the Principal Axes.*

159. The value of K_1 for the axes OZ_1 and OY_1.

Substituting the values of y_1 and z_1 from (157) in the expression

$K_1 = \int\int y_1 z_1 d^2A$, we have

$$K_1 = \int\int [y^2\sin\theta\cos\theta + 2y(\cos^2\theta - \sin^2\theta) - z^2\sin\theta\cos\theta]d^2A$$

$$K_1 = \sin\theta\cos\theta(I_z - I_y) + (\cos^2\theta - \sin^2\theta)K$$

Comparing the value of K_1 with the value of $\frac{dI_{z1}}{d\theta}$ we see that

$$\frac{dI_{z1}}{d\theta} = -2K_1$$

so that, when dI_{z1} is zero, $K_1 = 0$ and conversely.

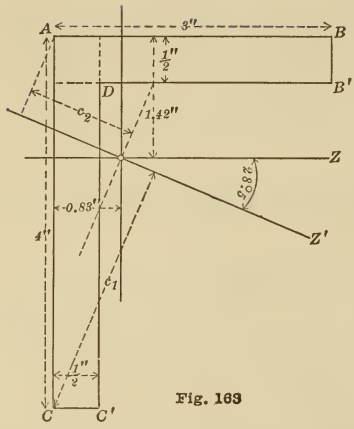


Fig. 163

Hence, the "Product of Inertia" is zero, for the Principal Axes, without regard to symmetry.

159. Angle irons. It is frequently necessary to know the maximum and minimum values of I_o for a cross-section of an "Angle Iron" shown approximately* in Fig. 163. It has no axis of symmetry, and the solution of an illustrative problem will be best with numerical values.

The centroid is found by taking moments of areas about AB and AC.

The centroid is distant from AB 1.42″, and from AC, 0.83″

The position of the centroid is shown at o.

160. How to find I_z, I_y and K. They can, of course, be found directly from the figure by means of the formula for parallel axes, but the following is more simple.**

1. Draw two axes, Z' and Z'', parallel to OZ, and thru the centroids of A_1 and A_2 respectively. Fig. 164.

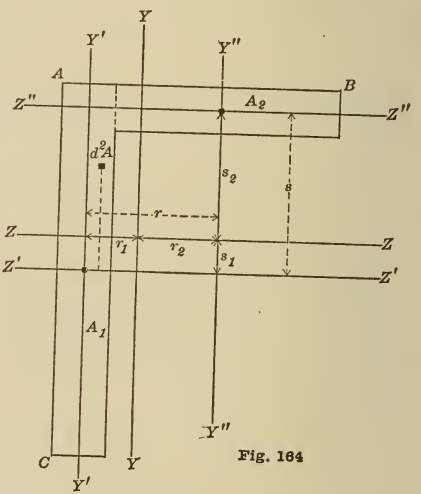


Fig. 164

Then $$I_z = I_{z1} + A_1 s_1^2 + I_{z''} + A_2 s_2^2$$

But $A_1 s_1 = A_2 s_2$, and $s = s_1 + s_2$,

hence $$s_1 = \frac{A_2 s}{A_1 + A_2}, \text{ and } s_2 = \frac{A_1 s}{A_1 + A_2}$$

so that $$A_1 s_1^2 + A_2 s_2^2 = \frac{A_1 A_2^2 s^2 + A_2 A_1^2 s^2}{(A_1 + A_2)^2} = \frac{A_1 A_2 s^2}{A_1 + A_2}$$

and $$I_z = I_{z'} + I_{z''} + \frac{A_1 A_2 s^2}{A_1 + A_2} \qquad (1)$$

In the same way it is readily shown after drawing the axes Y' and Y'' that

$$I_y = I_{y'} + I_{y''} + \frac{A_1 A_2 r^2}{A_1 + A_2} \qquad (2)$$

---

* In angle irons as actually used, the salient angles B' and C' are rounded, and the re-entrant angle D has a fillet. See Fig. 147.

** This elegant analysis appears to be due to Muller-Breslau.

2. To find $K=\int\int yzd^2A$, we take the parts A_1 and A_2 separately so that $K=K'+K''$. We see that

$$y=y'-s_1$$
$$z=z'-r_1$$

and $$yz=y'z'-r_1y'-s_1z'+s_1r_1$$

hence $$K'=\int\int y'z'd^2A-r_1\int\int y'd^2A-s_1\int\int z'd^2A+s_1r_1\int\int d^2A$$

But the axis $Z'Z'$ is an axis of symmetry of A_1, hence

$$\int\int y'z'd^2A=0$$

As the axes z' and y' pass thru the centroid of A_1, we have $\int\int y'd^2A=0$, and $\int\int z'd^2A=0$; and since $\int\int d^2A_1=A_1$ we have $K'=s_1r_1A_1$ and similarly $K''=s_2r_2A_2$

But, as before $$s_1=\frac{A_2s}{A_1+A_2},\text{ and } s_2=\frac{A_1s}{A_1+A_2}$$

$$r_1=\frac{A_2r}{A_1+A_2},\text{ and } r_2=\frac{A_1r}{A_1+A_2}$$

hence $$K=K'+K''=\frac{A_1A_2^2sr}{(A_1+A_2)^2}+\frac{A_1^2A_2sr}{(A_1+A_2)^2}$$

and $$K=\frac{A_1A_2sr}{A_1+A_2} \tag{3}$$

3. The numerical values of I_z, I_y and K are now easily found. From Fig. 163 we have $A_1=2$, $A_2=\frac{5}{4}$

and from figure 164

$$s=\left(2-\frac{1}{4}\right)=\frac{7}{4}\text{ and }r=\frac{1}{4}+\frac{5}{4}=\frac{3}{2}$$

Hence

$$I_z=\frac{\frac{1}{2}(4)^3+\frac{5}{2}\left(\frac{1}{2}\right)^3}{12}+\frac{2\times\frac{5}{4}\times\frac{49}{16}}{2+\frac{5}{4}}$$

$$I_z=2.696+2.356=5.052$$

$$I_y=\frac{4\left(\frac{1}{2}\right)^3+\frac{1}{2}\left(\frac{5}{2}\right)^3}{12}+\frac{2\times\frac{5}{4}\times\frac{9}{4}}{2+\frac{5}{4}}$$

$$I_y = 0.692 + 1.731 = 2.423$$

$$K = \frac{2 \times \frac{5}{4} \times \frac{21}{8}}{2 + \frac{5}{4}} = 2.02$$

4. It will be noted that all these quantities, tho they refer to axes meeting in the centroid of the surface, do not involve a knowledge of where that centroid is. The quantities s and r are the co-ordinates of the centroid of A_2 with reference to the axis meeting at the centroid of A_1. When, however, we calculate the angle θ for a principal axis, and proceed to draw that axis, the centroid of the entire surface must be used as found above.

The angle θ is determined by formula

$$\tan 2\theta = \frac{-2K}{I_z - I_y} = \frac{-4.04}{5.06 - 2.42} = -1.53$$

$$2\theta = -5.6^\circ \ 50'$$

$$\theta = -28^\circ \ 25'$$

Having the numerical values of I_z, I_y, K, θ and the co-ordinates of the centroid of the entire surface, we can plot the center, draw the principal axes and compute the values of $I(max.)$ and $(J\ min.)$.

5. The computation is somewhat simplified by combining equations (1) and (2) **157** , with the equation

$$\tan 2\theta = \frac{-2K}{I_z - I_y}$$

and we get

$$I = I_z - K \tan \theta^* \tag{4}$$

Combining this last equation with $I + J = I_z + I_y$, we get

$$J = I_y + K \tan \theta \tag{5}$$

---

* Eq. 4, **157**, gives

$$I_y = I_z + \frac{2K}{\tan 2\theta} = I_z + \frac{2(\cos^2\theta - \sin^2)K\theta}{2 \sin\theta \cos\theta}$$

Substituting in (1) we get

$$I = I_z - K\left(2 \sin\theta \cos\theta - \frac{\sin\theta\,(\cos^2\theta - \sin^{\ 2}\ }{\cos\theta}\ \right)$$

$$= I_z - K\left(\frac{2 \sin\theta \cos^2\theta - \sin\theta \cos^2\theta + \sin^3\theta}{\cos\theta}\right) = I_z - K \tan\theta$$

Since $K=2.02$ and $\tan(-28°\ 25')=-0.541$, we have

$$I=6.145 \qquad k_1^2=1.88$$
$$J=1.33 \qquad k_2^2=0.41$$

since $A_1+A_2=\frac{13}{4}$, and $k^2=\frac{I}{A_1+A_2}$, k being the radius of gyration.

161. When the angle-iron which we have been studying is used as a beam, the axis for I should be horizontal, and the distance to the extreme fiber should be computed or measured from an accurate drawing. See Fig. 163. Then the greatest Moment of Resistance the bar is safely capable of is

$$M_{max}=\frac{p_1}{c_1}\cdot A\times k_1^2=\frac{p_1}{c_1}\cdot I \qquad (1)$$

However, if an angle-iron is used as a strut of post, it is liable to bend by buckling, and as the buckling is always in the plane of *least resistance*, it is only the *least* Moment of Resistance which is to be counted upon.

$$M_{min}=\frac{p_1}{c_2}\cdot Ak_2^2=\frac{p_1}{c_2}J \qquad (2)$$

Steel "angles" may be rolled of various dimensions, with some allowances for round corners, and a fillet, but, in fact, only certain standard sizes are in the market. A general formula for k_1 and k_2 is hardly worth while here.

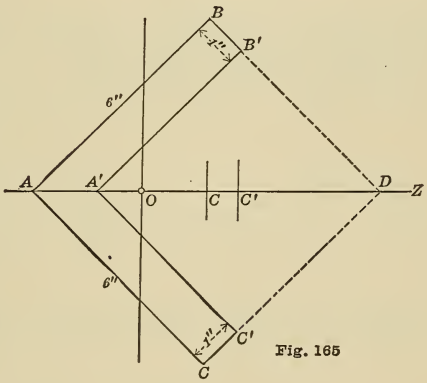


Fig. 165

162. When the arms of the angle iron are equal, as in Fig. 165, there is an axis of symmetry, AD. Consequently, $K=0$, and AD and the perpendicular thru the centroid of the surface are the principal axes. The Moment of Inertia, I_z, is readily found by subtracting the moment of inertia of the triangle $A'B'D$, with reference to the axis OZ, from the moment of inertia of the triangle ABD, and doubling the result. The moment of inertia, I_y, is found by subtracting the moment of inertia of the small square from the moment of inertia of the large square, after finding the centroid of the surface.

The distance $\overline{OC}$ is found from the equation

$$(\overline{OC})[(6)^2-(5)^2]=(5)^2\sqrt{\tfrac{1}{2}}=(5)^2(\overline{CC'})$$

Using the dimensions shown in the figure, find I_z and I_y.

163. A Z-bar. A Z-bar usually has equal flanges, and tho the cross-section has no axis of symmetry, its centroid is readily seen. With the axis and dimensions as shown in the figure (Fig. 166), find the Principal Axes and the max. and min. values of $M = aI$.

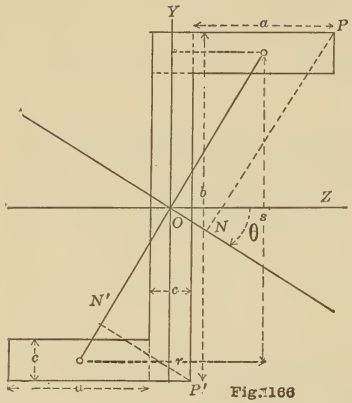


Fig. 166

$$I_z = \frac{cb^3}{12} + 2\left(\frac{ac^3}{12} + ac \times \left(\frac{b-c}{2}\right)^2\right)$$

$$I_z = \frac{cb^3 + 2ac^3}{12} + \frac{1}{2}ac\,(b-c)^2$$

$$I_y \frac{bc^3 + 2a^3c}{12} + \frac{ac}{2}(a+c)^2$$

$$K = \frac{1}{2}ac(b-c)(a+c).$$ See (3) **160.***

Hence $$k_z^2 = \frac{I_z}{A}, \text{ and } k_y^2 = \frac{I_y}{A}$$

Now, as $$\tan 2\theta = -\frac{2K}{I_z - I_y}$$

the angle θ, which is a negative angle, is readily found, as are the principal moments of inertia:

$$I_1 = I_z - K \tan\theta$$

$$I_2 = I_y + K \tan\theta.$$

Example.

Let $a = \frac{11}{4}$, $b = 6$, $c = \frac{3}{4}$, inches, so that $A = A_1 + A_2 + A_3 = 8.625$

Then $$I_z = \frac{1}{12}\left(\frac{3}{4} \times 216 + 2 \times \frac{11}{4} \times \frac{27}{64}\right) + \frac{1}{2}\left(\frac{11}{4} \times \frac{3}{4} \times \frac{441}{16}\right) = 42.12$$

$$I_y = \frac{1}{12}\left(6 \times \frac{27}{64} + 2 \times \frac{1331}{64} \times \frac{3}{4}\right) + \frac{1}{2} \times \frac{11}{4} \times \frac{3}{4} \times \frac{49}{16} = 15.44$$

$$K = \frac{1}{2} \times \frac{11}{4} \times \frac{3}{4} \times \frac{21}{4} \times \frac{7}{2} = 18.95.$$

---

* $K = \frac{A_1 A_2 sr}{A_1 + A_2}$. $A_1 = A_2 = ac$, $s = b - c$, $r = a + c$.

Hence $$k_z^2 = \frac{42.12}{8.625} = 5.00, \text{ so that } k_z = 2.24$$

and $$k_y^2 = \frac{15.44}{8.625} = 1.79, \text{ and } k_y = 1.34$$

$$\tan 2\theta = -1.42$$

$$2\theta = -54°51'$$

$$\theta = -27°25'$$

$$\tan\theta = -0.519$$

$$K\tan\theta = -9.835$$

$$I_1 = 42.12 + 9.835 = 51.96$$

$$I_2 = 15.44 - 9.835 = 5.60$$

so that $$k_1^2 \text{ max} = \frac{51.96}{8.625} = 6.02$$

and $$k_2^2 \text{ min} = \frac{5.60}{8.625} = 0.649$$

Hence $$k_1 = 2.45, \text{ and } k_2 = 0.81.$$

164. The relation of M to I. In computing the strength (moment of resistance) of a beam, it must not be forgotton that it is *not enough* to know the moments of inertia. Suppose we have, as is the case of the Z-bar, the max. I_1 and the min. I_2. The ratio of strength is

$$\frac{M_1}{M_2} = \frac{\frac{p_1}{c_1} I_1}{\frac{p_1}{c_2} \cdot I_2} = \frac{c_2 I_1}{c_1 I_2}. \tag{22}$$

This shows that the strength varies *directly* as the I, and *inversely* as the extreme fiber distance. For the Z-bar

$$\frac{M_1}{M_2} = \frac{I_1}{I_2} \cdot \frac{P'N'}{PN} \cdot = \frac{51.96}{5.60} \cdot \frac{P'N'}{PN}.$$

Now, from Fig. 166, $$PN = \frac{b}{2}\cos\theta + \left(a + \frac{c}{2}\right)\sin\theta = 4.10$$

$$P'N' = \frac{b}{2}\sin\theta + \left(a + \frac{c}{2}\right)\cos\theta = 1.71$$

Hence

$$\frac{M_1}{M_2} = \frac{51.96}{5.60} \cdot \frac{1.71}{4.10} = \frac{88.85}{22.96} = 3.9 \text{ nearly.}$$

165. *Z*-bars are used as purlins in the framing of roofs, the flanges taking the slope of the roof. *Z*-bars are used in the construction of columns and struts, especially where it is necessary to have ready access to all surfaces for the purpose of painting. The method of using them is shown by Fig. 167, which gives a general horizontal section.

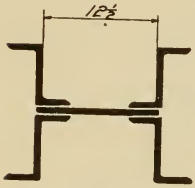


Fig. 167

The combination of 4 *Z*-bars, united by an interior plate, makes an open column whose principal moments of resistance are readily calculated, as the axes are parallel to those already used. The central connecting plate has usually the same thickness as the *Z*-bars.

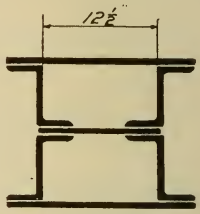


Fig. 168

In large columns, two wide thick plates are riveted to the outer flanges of the *Z*-bars, as shown in the drawing, Fig. 168. There is, however, a slight lack of economy in the use of material as compared with some other forms.

CHAPTER XI.

Elementary Graphical Statics.

166. Direct stresses in the members of a frame.

1. The frames considered in this chapter are ideal; they are assumed to be rigid; all members lie in a vertical plane; all joints are by means of pins which are central to the connected bars; and the only stresses (unless otherwise stated) taken into account are the *direct stresses* due to concentrated imposed loads. All loads are applied at pins, and their vectors are *external* to the frames; that is, a load on a lower pin is represented by an arrow *below* the pin; if on an upper pin the arrow is *above* the pin. Every pin is in equilibrium under the action of a system of balanced forces, and in every case the stress diagram will show for the balanced forces a static polygon.

2. A bar in tension will be called a "tie"; one in compression will be called a "strut." It should go without saying that a *tie pulls* on *both* pins, and a *strut pushes* on *both* pins. All drawings of frames and diagrams should be executed on a drawing board, with good instruments and on a scale at least twice that used in this book. Great

pains must be taken in drawing parallel lines. In laying off load vectors and in measuring stress vectors, the same scale must be used, but the scale itself is entirely arbitrary, tons, lbs., kilos, poundals or dynes.

167. Review static polygons. At this point the student will do well to review (**70**) and make sure that he can draw the static polygon for a set of co-planar forces, whose lines of action meet at a point and balance, provided all directions are given and all magnitudes but two are known.

It is also necessary to recall the fact that the arrows on the lines of a static polygon show the directions in which the members meeting at a pin act upon it, so as to balance. For example, suppose the bars 1, 2, 3, 4, 5, Fig. 169, act upon a pin at P, and their actions are accurately shown by the force polygon which closes, as the arrows follow each other around the *Area P*. These arrows show that Nos. 1, 3 and 5 must be *struts*; and that Nos. 2 and 4 are *ties*. The student will do well to study the figure 169 carefully so as to see the *system* followed.

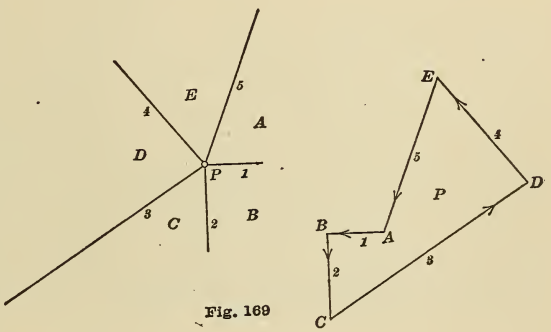


Fig. 169

168. Given an irregular cantilever frame supporting at different pins six equal loads. Fig. 170.

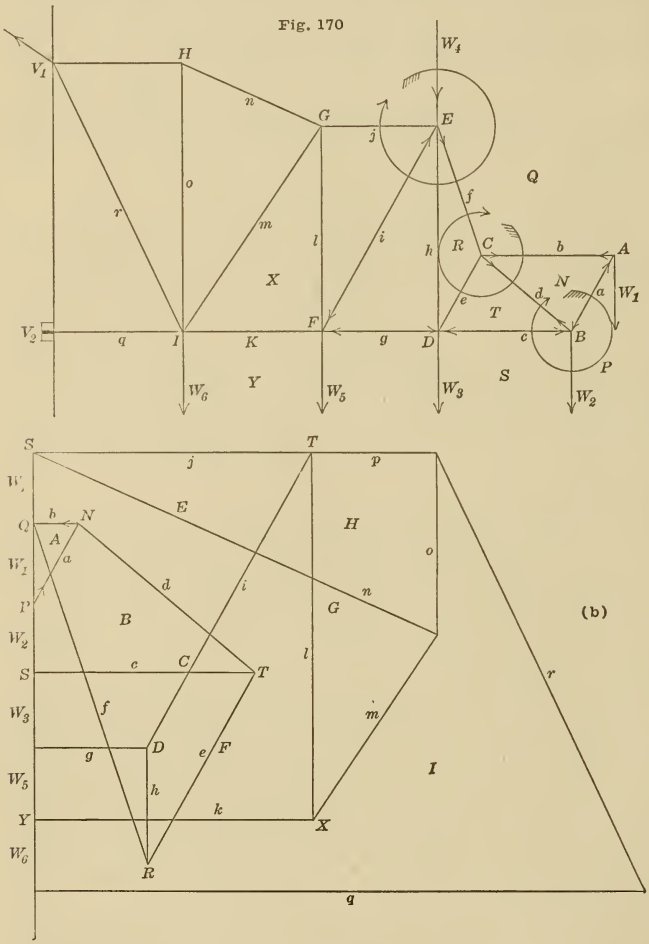
1. We begin with the pin A, because that is the only one whose static polygon (triangle) can at first be drawn. The forces are the weight W_1, and the actions of a and b; Fig. (b) shows the static triangle, QPN, whose area is also lettered A. Taking the forces in *clockwise* order round the pin, *i. e.*, W_1, a, b, the triangle shows that a acts *up against* the *pin*, and is therefore a strut; and that b *acts to the left, from the pin*, being therefore a tie.

2. The next pin is B acted upon by four forces: a already found to be a strut acting *down* on B; a known load, W_2, and the bars c and d. Taking the known forces in clockwise order, and using the stress line, NP, for a as drawn, we add $W_2 = PS$; then a line, ST, parallel to C, and get back to our starting point, N, at the upper end of a, by a line TN parallel to d. We thus have for the pin B the static polygon: $a - W_2 - c - d$, or $NPST - N$. The polygon should be clearly identified

no matter what other lines are in the neighborhood. It is seen that c is a *strut*, and d is a *tie*.

3. We next consider the pin C, acted upon by two known, and two unknown forces. Starting at Q, we trace $b-d$, and close back to Q by $e-f$. The reader must clearly identify the polygon $b-d-e-f$, or $QNTR-Q$, thinking of the *directions* of the actions as he goes around: e is *down*, and f is *up*. All the bars acting on the pin C are ties.

Fig. 170



4. Five forces act on D, of which three are known. Beginning with e at the point R, we retrace $e-c$, lay down W_3, draw g and h, and we have the polygon D. Remember that every static polygon must close.

5. In drawing polygon for E, we must begin with W_4, since the *known forces must be taken first in clockwise order*; and as f must *follow* W_4, the latter must be laid off *above* W_1; hence we have $W_4-f-h-i-j$, the last two being determined by required directions, and the necessity of returning to the point S.

6. The polygon for F is readily drawn, beginning and ending with the point T.

7. The four-sided polygon for G: $j-l-m-n$, being a quadrilateral with an internal intersection, is readily seen in Fig. (b).

8. The reader should re-draw the figure from the *frame* alone, identify every polygon, and note the nature of the stress in every bar. If he puts an additional load at G, he will find that the stress lines for p and j will not overlap.

169. A careful study of the above exposition will bring to the reader's notice several important relations between Fig. 170 and Fig. 170*b*.

1. We followed a *clockwise* order in considering the forces acting on a pin; as for example, the pin I; we had four known forces; so we took them thus: $o-m-k-W_6-q-r$: right-handed; in identifying the *polygon*, we took the stresses in the same order, but we went about the *area* I in a left-handed direction.

2. Not only does every pin or point of intersection in the frame have an *area* in Fig. (*b*), but every terminal meeting *point* in (*b*) has an *area* in Fig. 170. For example, take the point X, in (*b*) where the stress lines of l, m, k meet; in Fig. 170 the *bars* l, m, k surround the area X.

3. Again, we see that at the point Q, in Fig. (*b*), the force lines of W_4, b, f, and W_1 meet; now, turning to Fig. 170 we see that the *bars* f, b, and the indefinitely extended load lines, W_1 and W_4, bound, or shut off, the indefinite area represented by Q in Fig. 170. Similarly, the area Y, Fig. 170, corresponds to the meeting of the boundary lines W_5, k and W_6.

4. We note that the bar k, in Fig. 170, is the line separating the areas X and Y (like a division fence); it is evident that the bar k might be read: "the bar XY" (like the Mo.-Kansas line); and the stress in that bar as XY in Fig. *b*.

5. It will be seen that the external forces, *i. e.*, the loads W_4, W_1, $W_2, \ldots\ldots W_6$, all fall in the same vertical line in (*b*), and in regular order as one passes around the frame externally in a clockwise manner. The line of loads is called, "The Load Line."

6. If all the areas, within and without the given loaded frame, were suitably lettered, any pin could be identified by naming the areas around it; thus the pin A could be read: "the pin QPN," which is the way we read the static triangle of the forces at that pin.

The last two suggestions will be adopted hereafter in lettering frames and stress diagrams.

170. Method of lettering. A single example will show its advantages.

Fig. 171 shows a second example of an irregular cantilever frame with a definite load at each pin. All areas are lettered both within and without, and the order of procedure at every pin is indicated by a curved arrow. The stress diagram is shown in Fig. *b*. The

reader will readily see in Fig. 171 the bars CD and AH; and in (b) the magnitude of the stress in CD and again in AH. If he wishes to know the nature of the *stress* in CD, he must think of its action on *one* of the *pins*. If he thinks of the *upper* pin, he will read that pin (that is the areas around it) as $G-C-D-E-F$; now he will read the stress polygon also as $G-C-D-E-F$, and note that he read $C-D$ *downwards*; hence the action *on the upper pin is down*; so CD is a *tie*. Had he thought of, and read, the lower pin, he would have (mentally) said $D-C$, which reads *up* on the stress; so it pulls at the lower end as well. In the same way he finds that ED is a *strut*.

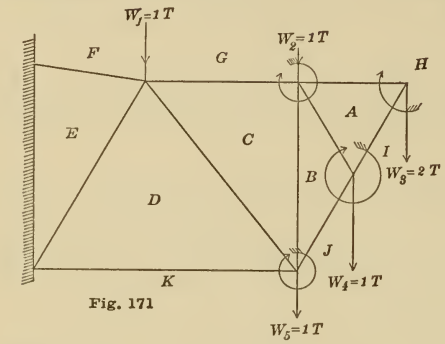
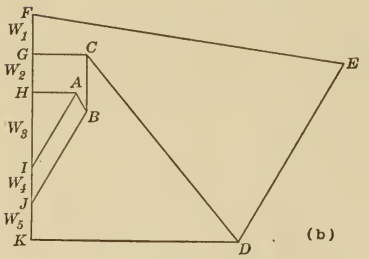


Fig. 171



171. Cantilever truss. Draw a stress diagram and scale off the stresses for the bars shown in Fig. 172.

172. Roof Trusses.

Ex. 1. *Plain Roof Truss*, with equal loads on all upper pins. Fig. 173. By symmetry the supports are equal. Directions: Draw first the Load Line, lay off the loads and supports to scale and letter them from K round to X. Next draw the static triangles, XAR and XKJ. Next, the static polygons $QRAB$ and $BAXC$ on the right, and $JKLI$ and $XJIH$ on the left. Finally, draw for the remaining upper pins. The symmetry of the figure will be an aid to accuracy. The many sided polygon for the central lower pin will be found already drawn.

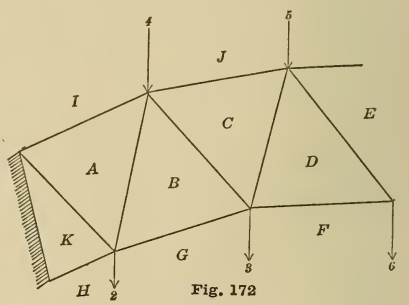


Fig. 172

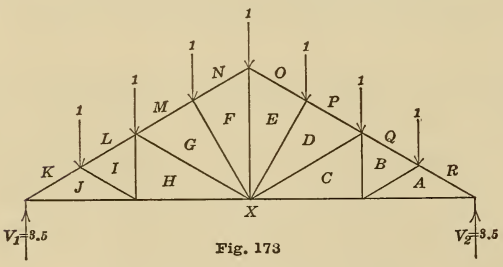


Fig. 173

The drawing will be finished when the character of every member is shown by the letter *s* (strut) or *t* (tie).

Ex. 2. *A Hip Roof.* Fig. 174. The points *A* and *D* in (*b*) coincide. Point out the polygon for the pin near *E* in the figure, 174 (*b*).

Ex. 3. Invent a roof truss having only triangular areas in the frame, and place on it symmetrical loads. Letter and draw the stress-diagram, and scale it.

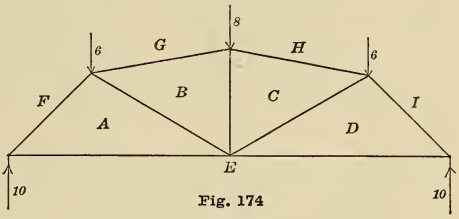


Fig. 174

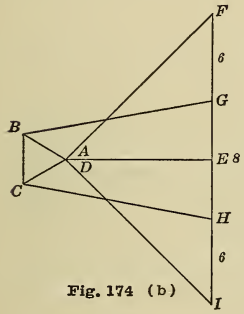


Fig. 174 (b)

173. Unstable frames. Before giving more problems, a word must be said about unstable frames.

1. Suppose a frame with pin joints is hinged at *X* and *Z* and loaded as shown in Fig. 175. [The reader of what follows *must draw the stress-diagram as he reads*, or he will not appreciate the thought.] To find the stresses on the assumption of stability, we lay off the loads W_1, W_2 and W_3; then begin with the pin *CDA*. We can next go to either the upper pin or the lower as in each case there are only two unknowns. Going to the upper, we say: *CABE*$-C$ and draw *AB* and *BE*. Now, going to the lower pin, we say: *BADF*$-B$; and the only unknown is *FB*; but, if it be drawn in the direction given, the *polygon will not close*; hence the frame is not *statical*, and *cannot support a load.*

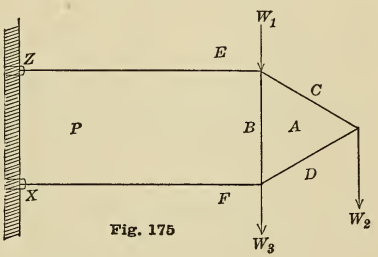


Fig. 175

If a diagonal bar from *Z* is put in, it will divide the area, and the polygon will close with a stress line *BP*, the length of which stress line shows *how much* a diagonal *tie* was needed. Equally well (and sometimes better) the other diagonal could be put in as a *strut.* It is thus seen that a four-sided area in a frame is *unstable* with pin joints. Even when the joints are nailed, screwed, glued, bolted or pinioned, they lack the stability which a diagonal strut or tie readily gives. A diagonal wire with its ends well secured is better than much nailing and glueing. Notice the weakness of screen frames, tables and benches when lateral forces act, and the sagging of doors and gates without diagonals when heavily loaded.

2. A rectangular frame without a diagonal is made rigid only by a large expenditure of material and labor on the joints. The economy and safety gained by the use of struts or metallic ties are worthy of more consideration by carpenters and cabinet-makers. Note the stiffness of aeroplanes. A four-sided frame often may profitably have two diagonal ties. A tie rod, wire or bar is a very simple affair, requiring little material and little workmanship as compared with a strut.

174. Queries to be answered by drawings.

1. If only a diagonal tie is put in, would the tension in it be greater or less than the compression in the strut, had the other diagonal been used?

2. Suppose the diagonals are of different lengths, the area B not being rectangular? Answer as above.

3. Suppose *both* diagonals are put in, which would take the stress? This is rather "indeterminate"; generally the tie or the one with the closest fittings. This query is answered later under **Deformations,** if the fittings are perfect and independent, and all bars are elastic, with known dimensions, and the strut does not buckle.

4. It was seen a few lines above that the diagonal inclined *downward towards* the support was a strut, while that inclined *upward towards* the support was a tie. When a vertical frame is subject to horizontal forces (in its plane) which may act either way, two diagonal ties suffice for both contingencies, and are vastly simpler than a single bar built to act as either tie or a strut as may be required.

Fig. 176 shows a rectangular screen frame, exposed to the action of horizontal forces tending to rack the joints.

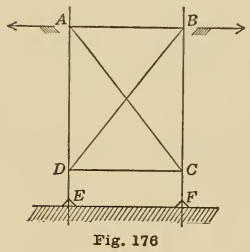


Fig. 176

When the force at B is acting, the tie, BD, supports the frame; when the force at A is acting, AC is in tension. In either case, if the feet are blocked, the action upon the parts DE and CF causes them to bend, and develop stresses which will be discussed later on.

175. Roof trusses under symmetrical loads.

1. By means of "purlins" (girders which cross from rafter to rafter), the loads on large roofs are brought to the trusses at the joints, or at supported points which *might* be joints. *All loaded points,* therefore, may be supposed to be joints with pins.

2. Draw the stress diagram for the loaded roof shown in Fig. 177, and mark the character of each member. Though the rafters may be continuous from eaves to ridge, they may be supposed to have

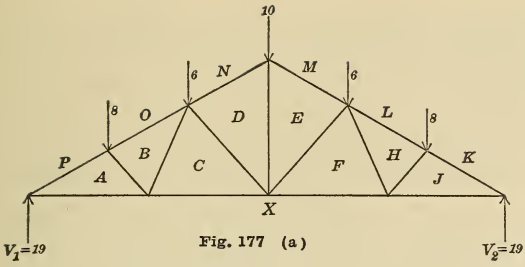


Fig. 177 (a)

pins where the loads are applied. The stress in JK is found to be greater than in EM.

3. Figures 178 and 179 are possible designs of a roof truss, which the reader may load, letter and draw.

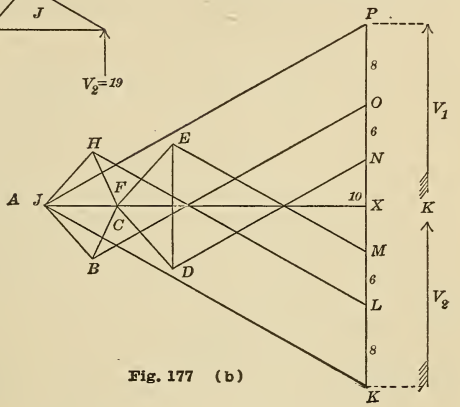


Fig. 177 (b)

176. Loads both above and below. A roof truss is often required to carry loads suspended from the lower pins. In that case the closed polygon of loads and supports (external forces) is a vertical line, some parts of which are used four times.

For example, take the simple truss, Fig. 180, in which the lower ties

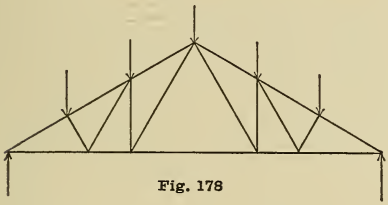


Fig. 178

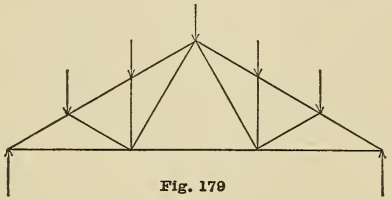


Fig. 179

(called the lower chord) are arranged to give more head-room below as shown in the sketch. The "load line" (the external forces taken in order around the truss, beginning at F) should be drawn first. The succession of forces is shown by lines not quite vertical on the right of the proper load line. Fig. 180*b*, on next page.

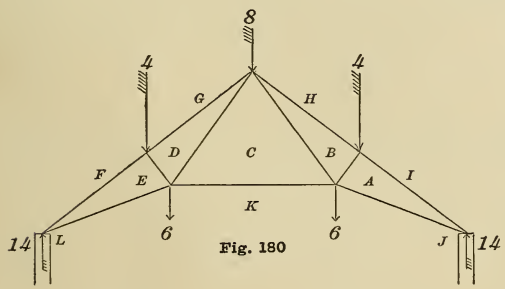


Fig. 180

177. A Fink truss. It often happens that a frame, or truss, has its long struts trussed (to prevent bending or sagging) by what may be called secondary trusses, thereby creating what to a novice seems an impossible case; this is well illustrated in what is

called the "Fink Truss." Fig. 181. It is seen that the rafters, after having been bisected, are again sub-divided by small secondary trusses, tho they are not wholly independent. For the sake of simplicity, uniform loads are given. There is no difficulty with three pins at either end. On going to a fourth pin, one finds both roads blocked, each pin having three unknowns. The difficulty may be overcome in different ways, but the simplest will be for the student to calculate the stress in XG by the method of moments. The whole trussed rafter to the right support is loaded with $30T$ (centered at the middle pin), and is held in equilibrium by the *support*, by XG, and by an action at the upper end, Z. Taking the upper end, Z, as an axis of moments, we have the moment equation, calling the span s,

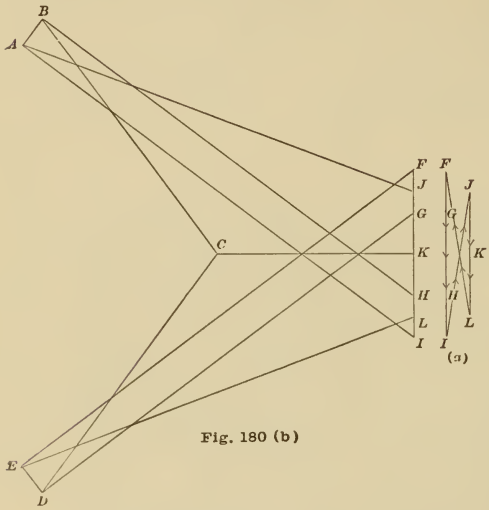


Fig. 180 (b)

$$M_z = \frac{30s}{4} - \frac{35s}{2} + XG.h = 0.$$

If the height $h = \frac{s}{2\sqrt{3}}$,

we find that

$$XG = 20\sqrt{3} = 34.64,$$

which is laid off to scale from X, Fig. 181(b) determining the point G on the diagram. Now take the pin $DCXG$. The rest of the problem is easy.

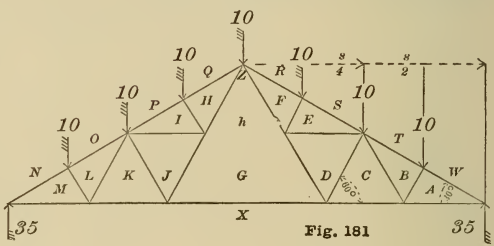


Fig. 181

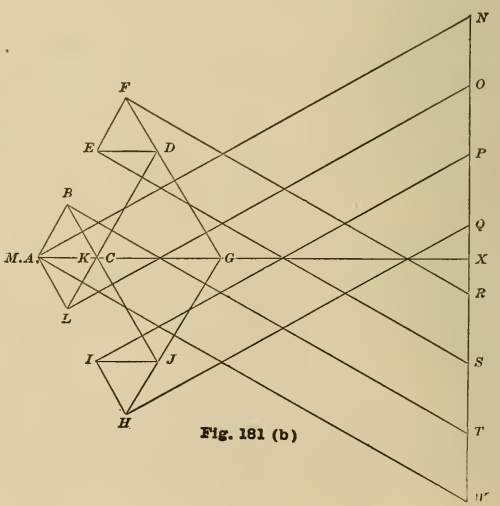


Fig. 181 (b)

178. Unsymmetrical loads. 1. Vertical loads

directly over supports do not affect the stresses in the members of a truss, hence they do not appear in the diagram. Pin loads are rarely equal, and it is found that when the interior members (the "bracing") are most severely stressed, the loads are not symmetrical. In such cases the supports are unequal and should be calculated (most readily by the method of moments), as already illustrated in Chapters III and V. See Fig. 182.

To find the support at one end, take a moment axis at the other end. In getting moments, use any convenient unit of length; the absolute length of a truss is of no importance.*

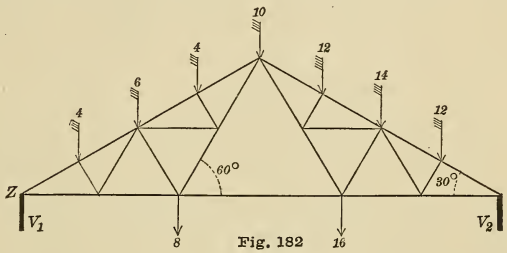


Fig. 182

2. Ex. Let Z be the axis for finding V_2. The upper joints divide the span into eight equal parts; the lower joints into three equal parts. Hence, let the span be 24 units; then we have for finding V_2

$$24V_2 = 12+36+36+120+180+252+252+256+64$$

$$V_2 = 50\tfrac{1}{3}, \text{ and consequently } V_1 = 35\tfrac{2}{3}.$$

The stress diagram will be found to be unsymmetrical.

3. There are other ways of finding V_1 and V_2, chiefly by drawing, but none are more simple or more readily used; even the above solution may be made more simple (and the student should simplify every mathematical and mechanical process) by noting that $12 = 4+8$; $14 = 6+8$, and $16 = 8+8$, so that the loading is equivalent to a symmetrical load which gives $27T$ on each wall, a load of $24T$ on the center pin of the right-hand rafter, and a load of $8T$ on the right-hand lower pin. The right-hand support, V_2, carries $\frac{3}{4}$ of the $24T$, and $\frac{2}{3}$ of the $8T$. So that the whole support on the right is

$$V_2 = 27+18+5\tfrac{1}{3} = 50\tfrac{1}{3}$$

all of which is seen without putting pencil to paper. (The student should go over this paragraph till every step is clear.)

179. Inclined loads, wind pressures. When load lines are not vertical, the supporting forces, one or both, are not vertical, and the external forces acting on a truss form an *open* static polygon. Some-

* Given the concentrated loads, the mere size of a truss has nothing to do with the direct stresses in its members, tho it has much to do with the way in which the members shall be given sufficient strength and stiffness.

times the stresses due to vertical loads and those due to inclined loads (the wind) are found separately and added. The following illustration takes them all together.

1. Steel trusses are generally anchored firmly to the windward wall (on the side which is exposed to the highest winds), while the free ends are either resting on rollers, or placed in grooves or shoes which admit of an easy horizontal motion. The end is left free so that the truss may expand or contract as the temperature rises or falls, or lengthen and shorten again when loads are put on and taken off, without moving the walls. Accordingly the supporting force at the free end is vertical, or nearly so.

It is usually assumed that wind pressure is normal to the surface of action, and that it is uniformly distributed. Neither assumption is very near the truth.* The wind pressure per square foot is a very uncertain quantity. It depends chiefly on the velocity of the wind and the slope of the roof; and somewhat on the extent of surface. Its action is concentrated at the joints as is the "dead" load.

2. *To find the stresses in the members of the roof truss shown in Fig.* 183, *which carries dead loads, and wind pressures.* The support, V_2, is found by getting the moment of the wind pressure about an axis at Z, the fixed end, and adding the support due to the symmetrical dead load. [The drawings are on the opposite page.]

The support due to the dead load is 12 tons.

The total wind pressure is 36, and its "center" is at the middle of the rafter. If r be the length of the rafter, the wind's moment is $18r$. Let the inclination of rafter be $30° = \beta$.

Then the support, V_2' due to the wind, at the wall on the left, has a moment arm, $2r \cos \beta$; hence we have the equality of moments

$$2V_2'r \cos \beta = 18r$$

$$V_2' = \frac{9}{\cos \beta} = 6\sqrt{3} = 10.39$$

and

$$V_2 = 12 + V_2' = 22.39$$

Now, the static polygon for external forces can be drawn, beginning with XJ, which is 22.39. Since the polygon *must close*, the last line, SX, shows the magnitude and direction of the support V_1. The builder must see that this support is furnished. The effect of the wind is plainly shown in the stresses of rafters and bracing on the windward side, and the chords XA and XC.

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* See a paper on Wind Pressure by Prof. F. E. Nipher, Proceedings of St. Louis Academy of Science.

180. Supports modified by wind pressures. The student who has read Chapter V will recall that the resultant of vertical and inclined loads could have been found graphically by means of the *Chain Polygon*, and that the components of the force balancing it, *i.e.*, V_1 and V_2, could also have been found by the method there explained. However it is doubtful if there is a simpler solution than that just given, but the student should not assume that there is no other method.

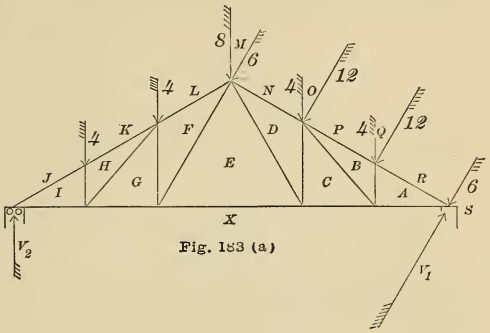

Fig. 183 (a)

Ex. Let the student add wind pressures to the load on the truss shown in Fig. 177, assuming that the windward support is anchored, while the other rests on rollers.

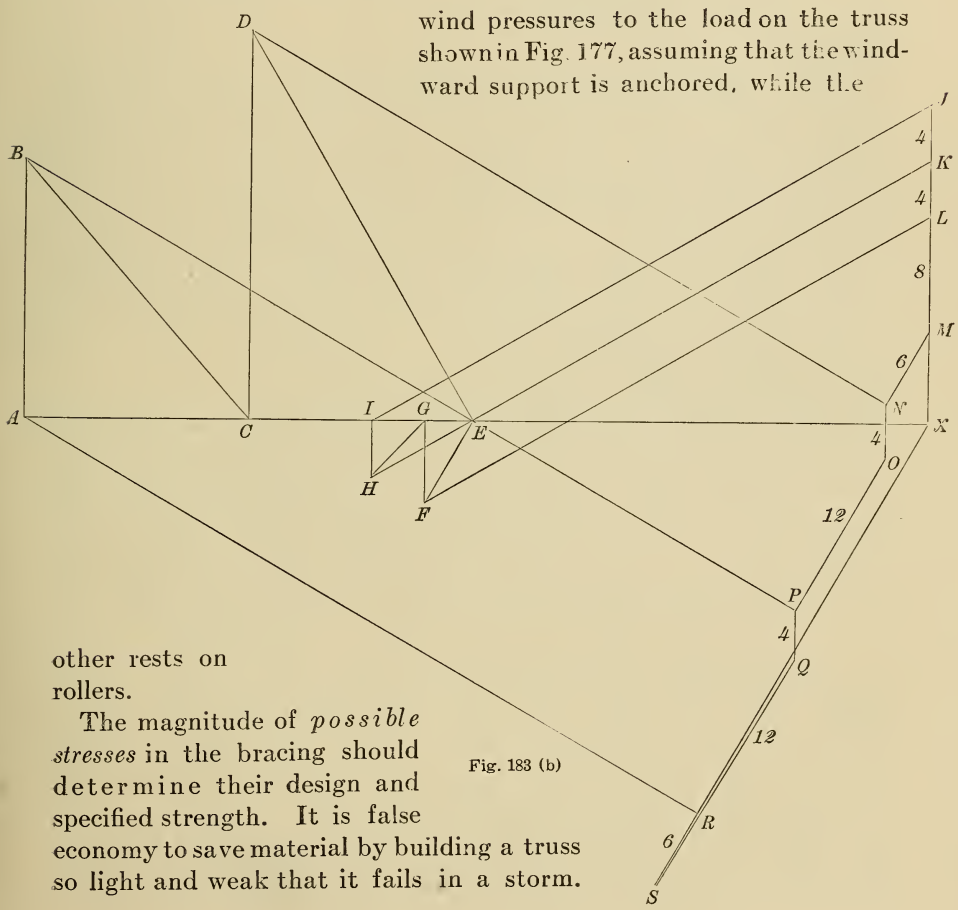

Fig. 183 (b)

The magnitude of *possible stresses* in the bracing should determine their design and specified strength. It is false economy to save material by building a truss so light and weak that it fails in a storm.

Bridge Trusses.

181. The two trusses of a bridge are connected by cross-girders which usually rest upon the pin joints of the trusses. These girders carry stringers, and the stringers carry the roadway; hence the loads are applied to the trusses at the joints. The weight of a member itself is divided equally between its ends. Bridges may have "thru" or "trellis" spans for moving loads to pass between the trusses, or "deck" spans when the roadway and its immediate supports are above the trusses. The stresses in members are readily found graphically, the results may be checked by other methods.

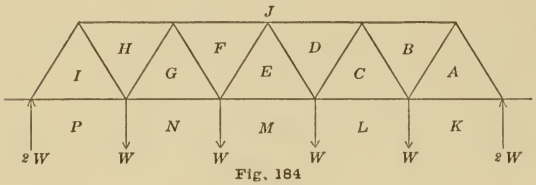

Fig. 184

182. A "Warren Girder," or triangular truss. Fig. 184. The triangles are equilateral, and only loads on the lower pins are considered. The load line is, in order,

$$+JK-KL-LM-MN-NP+PJ.$$

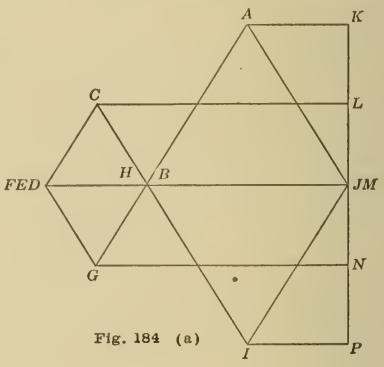

Fig. 184 (a)

The fact that the letters D, E and F fall together shows that, under a symmertical load, the bars DE and EF have no stress. This is thus explained: the stress in FE (tension), caused by the load MN, is exactly cancelled by the compression caused by the equal load LM. Were the loads NM and ML unequal, both FE and ED would be under stress.

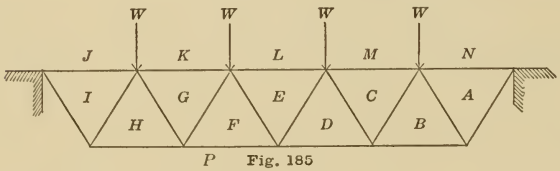

Fig. 185

2. **A deck span.** The supports are each $2W$, and the load line begins $+PJ$. Fig. 185. With this ideal loading the bars DE and EF are idle, but such would not be the case with irregular loading. Had the bridge been one panel longer, there would have been no idle member. Make a drawing to confirm this.

3. Draw a stress diagram for a thru span, and one for a deck span, with eight or ten panels under uniform loads. Mark struts and ties, and take note of laws of increase and decrease of stress in members.

183. Combination trusses. When panel lengths are too great for roadway stringers, two Warren girders are sometimes combined, as shown below. One rests on masonry, as in Fig. 184, and the other hangs on piers of steel or masonry as in Figure 185. Such a combination is known as a **Lattice Girder.**

2. Draw a diagram for the suspended truss with its *six* loads, and a separate diagram for the other with its *five* loads. Fig. 186.

Scale off all the stresses for the different members, and *add the stresses in the chords* wherever they have been used in both trusses. Finally write the stresses found on the *members themselves*, as represented in the drawing.

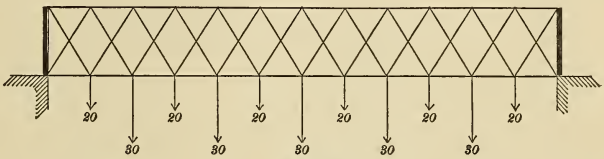

Fig. 186

It will be seen that the end panels of both chords are used but once. Let the component girders be drawn on good paper with an actual span of 12 inches.

3. Lattice girders are often met with which have three or four component trusses. In such cases some of the end members are inclined at special angles. See Fig. 187.

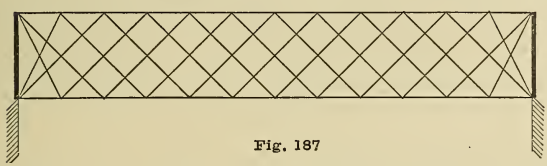
Fig. 187

184. Two typical bridge trusses are shown in their simplest form in order that the student may see their characteristic features, and by personal experience may learn the beauty and simplicity of the graphical analysis when one has good drawing instruments at hand.

1. **The Pratt Truss,** shown in Fig. 188.

2. The "Double Pratt" has short vertical struts and long diagonal ties. The two trusses have the same chords and end struts, but separate diagonals. The load on the vertical rod, near the end, is divided, usually, half and half. See Fig. 189.

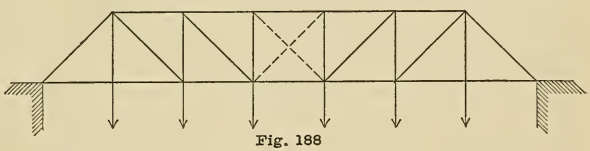
Fig. 188

Draw the diagrams separately, measure stresses and where the members are superposed, add the stresses.

185. A Howe truss. If a load is perfectly symmetrical (and it rarely is), the dotted members, which are both ties, will not be needed; with a moving load both are needed, but not both at once. The student will note the long diagonal *struts* in the "Howe," and the long diagonal *ties* in the "Pratt." The "Howe" went out of use when steel replaced wood in bridge construction. Fig. 190.

Fig. 189
A DOUBLE PRATT

For a full discussion of "Framed structures" which the above few pages are intended to lead up to, the student is referred to Building Construction and Bridge Engineering.

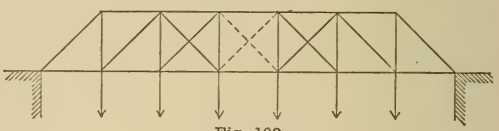

Fig. 190
A HOWE TRUSS

Calculations base on stress diagrams. In all the problems of the present chapter, the magnitude of stresses in frame members was determined graphically, and the degree of accuracy depended upon the quality of the instruments and the skill of the draftsman in laying off and measuring lines and angles. But it is easily seen that the diagrams, even when made "free-hand," lend themselves to trigonometric calculation. An examination of any one of the diagrams will suffice to discover how an unknown length can be computed, as all angles are known.

1. For example, let the student calculate the stresses in several members of the Fink truss, Fig. 181, and also of the Pratt Truss, Fig. 188, and compare results with his measurements of his stress-diagrams. He is quite likely to find defects in his instruments and faulty drawing.

2. Let the student make a stress diagram of the cantilever truss, Fig. 42.

186. The endless variety of designs. There is no limit to the number of frames for roofs and bridges. "King Post" and "Queen Post" roof trusses offer no difficulty, but no unusual design should be built without a careful drawing which shows both the *nature* and the *magnitude* of the stress in each piece. The writer once saw a roof truss for a church actually built with a tie (an eye-bar), where there should have been a strut. The builder's "horse-sense" was inferior to a stress diagram. A drawing of thetruss follows. Fig. 191.

The reader may assume any reasonable loading, and proceed with the drawing, using one of the members, *FG*; and note the stress in it. Then make a second diagram with the other diagonal, *FG*. See which would alone be a tie, and which alone would be a strut. The reason why the truss does not fall is, that the rafter is continuous to the ridge and very strong, and its *stiffness* saves it.

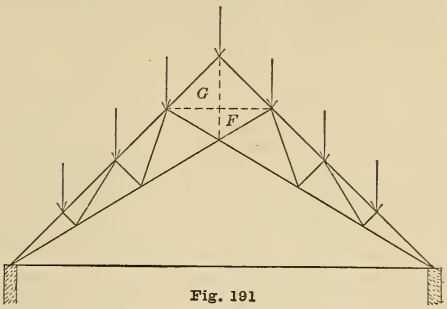

Fig. 191

CHAPTER XII.

Internal Stress.

Composition and Resolution of Stresses.

187. In Chapter IX it was shown that at every internal point of a loaded beam there was stress normal to a cross-section plane, and a shearing stress in the same plane.

The definition that "force is an action between two bodies," is to apply to two contiguous parts of a body.

Let *A*, *B* and *C* be parts of a horizontal loaded beam. Both *A* and *C* and the load Δw are acting on *B*, and holding it in equilibrium. There are actions at every point of the planes which bound *B*, and those actions vary from point to point.

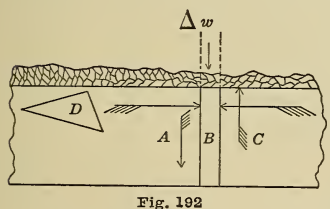

Fig. 192

But that is not all. If a prism like that at *D* be formed by three general planes cutting the same rectangular loaded beam, there is stress (actions) at every point of every lateral face. The adjacent material is pressing or pulling *D*, and is dragging or tending to drag it up or down; and yet, as *D* is at rest, we know that all such actions actually balance. The prism may be very small, infinitely small, yet have three sides or faces, and the stresses on those sides are normal, oblique and tangential. It is the purpose now to study the stresses on *every plane* that can pass thru a given point in a loaded beam, particularly planes which are perpendicular to the plane of external forces (loads and supports).

188. Simple stress and its components. Fig. 193 represents a bar in tension but under no other stress. If the *cross-sectional* area S is uniform, the "*Normal Intensity*" is

$$p_x = T/S$$

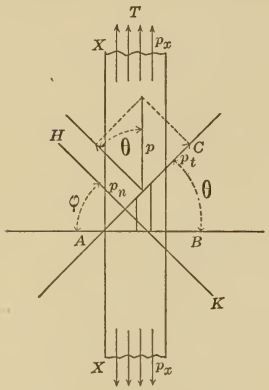

Fig. 193

upon every unit of surface of a right section.

If an oblique section, AC, is made, the stress is oblique, and as the section has the area $S_1 = S \sec\theta$, we have

$$p = \frac{T}{S \sec\theta} = p_x \cos\theta$$

If now p be resolved into its normal and tangential components, we have

$$p_n = p\cos\theta = p_x \cos^2\theta$$

$$p_t = p\sin\theta = p_x \sin\theta\cos\theta$$

If a second plane HK be taken so that its inclination to a right section is $\phi = \frac{\pi}{2} - \theta$, we shall have

$$p_n' = p_x \cos^2\phi = p_x \sin^2\theta$$

$$p_t' = p_x \sin\phi\cos\phi = p_x \cos\theta\sin\theta$$

Hence $p_n + p_n' = p_x$, and $p_t = p_t'$ if $\phi + \theta = \pi/2$.

This is particularly important, and is stated as follows: **If two oblique planes having their normals in the same axial plane are at right angles, the tangential components of the oblique stresses are equal.** A graphical representation of magnitudes of the stresses on an oblique plane is readily shown: The value of p_n is found by projecting p_x *twice* thru the angle θ. Fig. 194.

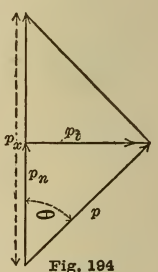

Fig. 194

189. Compound stress. More often than not, material is subjected to, or sustains, more than one simple stress. When the simple stresses are known, the stress on any given plane is readily found if all the normals lie in the same plane. This may be illustrated by *finding the stress on a third plane when the simple stresses on two planes are given.*

1. Take the case of **the shell of a steam boiler** or a cylindrical gas tank. Let the radius of the cylinder be r, the thickness of the shell be c, and the gauge pressure of the steam or gas per square inch be q in *excess of the external air-pressure.* The total direct longitudinal

tension is $q\pi r^2$ due to the pressures upon the ends of the cylinder, which must be sustained by the shell at every cross-section, whose area is $A = 2\pi rc$.

$$\text{Hence } p_y = \frac{q\pi r^2}{2\pi rc} = \frac{qr}{2c}$$

if the direct longitudinal tension is p_y.

2. In addition to this longitudinal tension, there is a "hoop tension" which is found by considering the tension in a hoop one inch wide around the cylinder. The resultant internal pressure on one half of the hoop is the same as the pressure would be on a *diametral* strip one inch wide; hence $2r \times q = 2T$, where T is the tension at every cross-section of the hoop. See Fig. 195.

$$\text{Hence } T = rq.$$

Now, the cross-section of the hoop is $c \times 1$, so that

$$p_x = T/c = rq/c$$

This shows that the *intensity of the hoop tension is twice* the intensity of the longitudinal tension.*

3. Unless otherwise stated as to the drawings which follow, the planes thru OX and OY are thought of as perpendicular to the plane of the paper, and intersecting in an axis OZ (not shown); and all oblique planes of action are parallel to OZ.

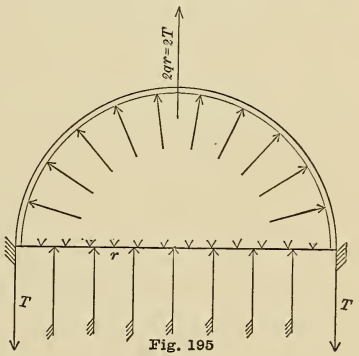

Fig. 195

4. It is now required to find the stress on the plane, or at the section AB, which makes an angle θ with OY. Fig. 196.

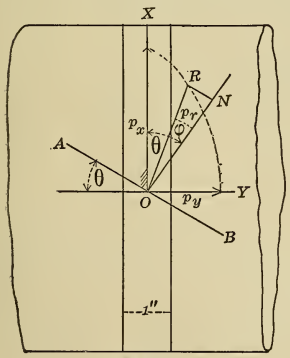

Fig. 196

Let ON be normal to the section. By the formulas just derived, resolving p_x into components on AB.

$$p_n' = p_x \cos^2\theta, \qquad p_t' = p_x \sin\theta\cos\theta;$$

Resolving p_y,

$$p_n'' = p_y \sin^2\theta, \qquad p_t'' = p_y \sin\theta\cos\theta$$

* The observing student will recall the fact that there are two rows of rivets along a longitudinal seam of a steam boiler, but only one row on a cross-sectional seam.

The magnitudes of p_n', p_n'', p_t', p_t'', etc., are found by projection in Fig. 197.

$$OX = p_x;\ OY = p_y;\ \overline{OC} = OC' = p_n';\ OD = OD' = p_n''.$$

$$ON = p_n = p_n' + p_n'';$$

$$NR = p_t' - p_t'' = p_t$$

$$OR = p_r$$

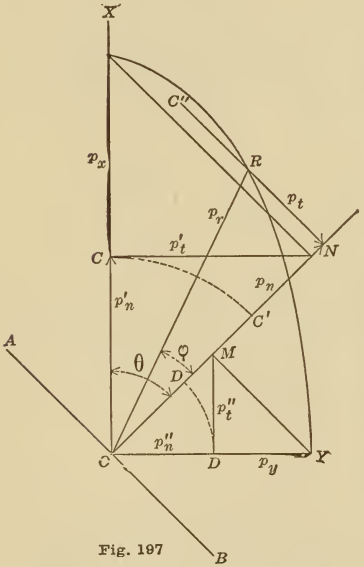

Fig. 197

From the figure it is seen that p_n' and p_n'' have the same sign, while p_t' p_t'' have opposite signs.

Hence we have the final equations

$$\left.\begin{aligned} p_n &= p_n' + p_n'' = p_x \cos^2\theta + p_y \sin^2\theta \\ p_t &= p_t' - p_t'' = (p_x - p_y)\sin\theta\cos\theta \end{aligned}\right\}$$

If we lay off p_n on ON, and p_t at right angles to ON on the *side* of the *larger* of p_t' and p_t'', then $p_r = OR$ **is the stress on the plane AB per unit of surface.** It is thus seen that the stress p_r is oblique, and that the obliquity, that is, the angle with the normal, is ϕ, and that

$$\tan\phi = \frac{(p_x - p_y)\sin\theta\cos\theta}{p_x\cos^2\theta + p_y\sin^2\theta}.$$

Substituting for p_x and p_y the values qr/c and $qr/2c$, we have

$$p_n = \frac{qr}{2c}(1+\cos^2\theta);\ p_t = \frac{qr}{2c}\sin\theta\cos\theta$$

$$\tan\phi = \frac{\sin\theta\cos\theta}{1+\cos^2\theta}$$

190. When the stresses on OX and OY are normal and known, it is much simpler to use the static triangle. Suppose we wish to find the stress on a general plane which is parallel to the inter-section of the planes OX and OY, and which makes an angle θ with OY. Fig. 198.

1. ON is normal to AB, hence $XON = \theta$. Take an elementary prism, whose faces are parallel to the planes AB, OX and OY, whose altitude is c, and whose bases have the edges $bk = dx$, $ak = dy$, $ab = ds$. It is evident that $dy = ds\cos\theta$, and $dx = ds\sin\theta$.

Now the face of the prism, $ak \times c$, has the stress p_x on every unit of surface; hence the stress on that face, which we will call $\overline{OE}$, and lay off on OX, is

$$\overline{OE} = cp_x \cos\theta \, ds$$

Similarly, the stress on the face, kb, is found to be

$$OD = cp_y \sin\theta \, ds$$

The resultant of these two stresses is

$$OR = cds \ (p_x^2 \cos^2\theta + p_y^2 \sin^2\theta)^{\frac{1}{2}}$$

If the intensity of stress on the third face is represented by p_r, the total stress on that face is $p_r cds$, but that stress must balance the resultant OR. Hence the *static triangle*, OER, and

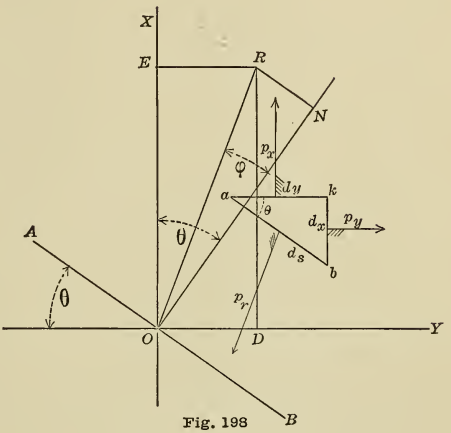

Fig. 198

$$RO = p_r cds,$$

$$p_r = (p_x^2 \cos^2\theta + p_y^2 \sin^2\theta)^{\frac{1}{2}}$$

This formula could easily have been found by reducing $p_r = (p_n^2 + p_t^2)^{\frac{1}{2}}$ in the former solution, **188.**

2. It will be noted that the height of our prism, and the hypotenuse ds have disappeared; only *unit* stresses remain. Hence, we will use only unit stresses, and in the place of $\overline{OE} = cp_x \cos\theta \, ds$, we will use $OF = p_x \cos\theta$. Fig. 199.

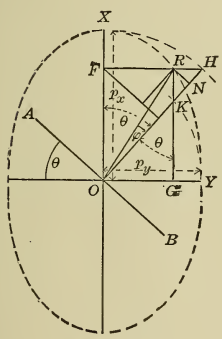

Fig. 199

If we lay off on OX $p_x \cos\theta = OF$, and on OY, $p_y \sin\theta = OG$, and draw the diagonal, **that diagonal, if drawn towards O, and read RO, will represent in *length* and *direction* the unit stress which *balances* the *stresses* p_x and p_y.** This statement is readily grasped if the student bears in mind that the stress of adjacent material on ds *must balance* the stresses on the faces dx and dy, since the prism is in equilibrium.

191. The ellipse of stress. Fig. 199. Let AB be the plane on which the stress is to be found. Draw the normal, ON, and lay off on it *both* $p_x = OH$ and $p_y = OK$, each to scale, and project them upon their respective axes, thus getting the co-ordinates of R and the magnitude of p_r to scale.

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In the case of the steam boiler under internal pressure, it was found that $p_x/p_y = 2$, but that relation was special; in general they may have any ratio. Whenever p_x and p_y are normal, and are the *only stresses* on their respective planes, which are perpendicular to each other, they are called the **principal stresses,** and their planes of action are called the **principal planes,** and the magnitude of p_r on any oblique plane lies between the magnitudes of p_x and p_y. From this construction it is evident that the locus of R as the plane AB changes its position, is an ellipse, whose semi-major axis is p_x, and whose semi-minor axis is p_y. If we call $OF = x$ and $OG = y$, we have $x = p_x \cos\theta$, and $y = p_y \sin\theta$; eliminating θ, we have

$$\frac{x^2}{p_x^2} + \frac{y^2}{p_y^2} = 1.$$

The figure is known as the *Ellipse of Stress.*

The "*obliquity*" of the stress is shown by ϕ, and the components of p_r, ON and NR, are

$$ON = p_n, \text{ and } NR = p_t$$

Their analytic values have already been found by projection;

$$ON = p_n = p_x \cos^2\theta + p_y \sin^2\theta, \text{ and } NR = p_t = (p_x - p_y)\cos\theta\sin\theta.$$

and

$$\tan\phi = \frac{(p_x - p_y)\sin\theta\cos\theta}{p_x\cos^2\theta + p_y\sin^2\theta}$$

192. Caution. Unless great care is exercised, the young student will overlook an important matter in thinking about finding the component stresses on an oblique plane, when p_x and p_y are given as the stresses on two rectangular planes.

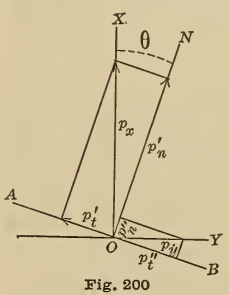

Fig. 200

The temptation is to resolve p_x into p_n' and p_t', and p_y into p_n'' and p_t'' as shown in Fig. 200, which is **all wrong.**

It is *not* the unit stress, p_x, on the plane OY, which is to be resolved into its components; it is *the unit stress on the plane AB which is to be resolved.* That stress is not p_x, it is $p_x \cos\theta$, and *that* is resolved into $p_x \cos^2\theta = p_n'$, and $p_x \cos\theta \sin\theta = p_t'$.

193. 1. **The practical value** of a problem like that of a boiler shell lies in the fact that it shows at a glance that when for any reason an oblique section in the shell is made, and the adjacent plates are connected by butt-straps, bolts or stays **the direction of the bolts or**

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